UNIVERSITY OF CALIFORNIA BERKELEY
Department of EECS

Problem 1: Sinc and Geometry

We know that if we have $x(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$, the CTFT of $x(t)$ is given by

$$X(\omega) = \begin{cases} 1 & |\omega| \leq \pi \\ 0 & |\omega| > \pi \end{cases}.$$

Suppose that we are instead interested in $y(t) = x^2(t) = \text{sinc}^2(t)$. Find and plot $Y(\omega)$.

Problem 2: A Very Normal Question

The Central Limit Theorem (CLT) states that $\sqrt{n}$ times the average of n independent and identically distributed (i.i.d.) zero-mean, unit-variance random variables converges to a standard Normal distribution as $n \to \infty$. In this question, we frame the CLT in the context of signals, using the fact that summing random variables corresponds to convolving their probability density functions (PDFs). We will prove the CLT using characteristic functions (the CTFT of a distribution's PDF) by showing that the infinite convolution of certain signals converges to a Gaussian.

Consider a non-negative, real-valued signal x whose spectrum $X = \mathcal{F}\{x\}$ has the following properties:

- $X(0) = 1$
- $X'(0) = 0$
- $X''(0) = -1$

(a) For some positive integer n, let $y_n(t) = \sqrt{n}\, x(\sqrt{n}t)$. Find Y_n, the CTFT of y_n, in terms of X.

(b) Now let's define the signal z_n to be n copies of y_n convolved together, i.e.

$$z_n(t) = \underbrace{(y_n * y_n * \ldots * y_n)}_{n \text{ times}}(t).$$

In the context of probability theory, this is the equivalent to taking $\sqrt{n}$ times the average of n i.i.d. random variables with PDF $x(t)$. So, by the CLT, $\lim_{n\to\infty} z_n(t)$ should be the PDF of the standard Normal distribution.

Determine a reasonably simple expression for Z_n, the CTFT of z_n.

(c) Let's explore what happens as we take more copies of y_n convolved together. Show that $\lim_{n\to\infty} Z_n(\omega) = e^{-\frac{\omega^2}{2}}$.

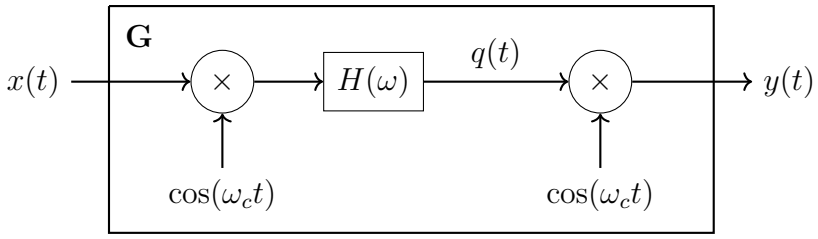
Hint: Consider evaluating $\lim_{n\to\infty} \ln Z_n(\omega)$ *instead. Alternatively, consider expanding* Y_n *into its second order Taylor expansion. You may find the following limit useful:* $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

(d) Determine an approximate expression for $z_n(t)$ when $n \gg 1$.

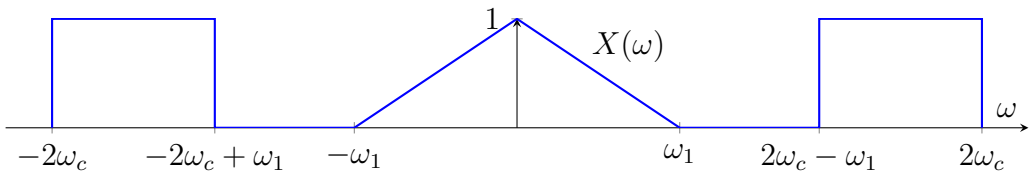
Hint: Recall the Fourier pair: $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\frac{\omega^2}{2}}$.

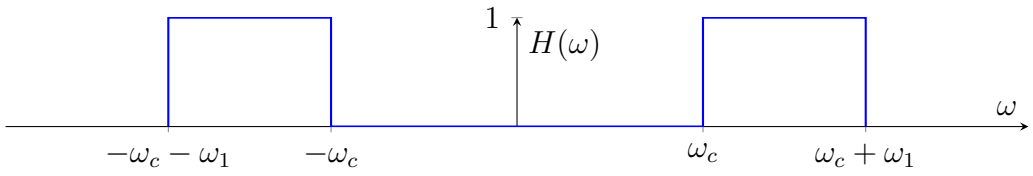
Problem 3: Modulation

A continuous-time system G consists of three processing blocks: modulation, followed by LTI filtering, followed by another modulation, as the figure below indicates.



The spectrum $X(\omega)$ of the input signal x, as well as the frequency response $H(\omega)$ of the LTI filter, are depicted below. Assume that ω_c is sufficiently large; in particular, $\omega_c > 2\omega_1$.





(a) Provide well-labeled plots of $Q(\omega)$ and $Y(\omega)$, the spectra of $q(t)$ and $y(t)$, respectively.

(b) Determine a reasonably simply closed-form expression for $h(t)$, the impulse response of the filter.