Problem 1: A horse walks into a system, it's a stable stable.

The following are the impulse responses of discrete-time LTI systems. Let $u[n]$ denote the unit step function. Determine whether each system is causal and/or stable.

(a) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[n-1]$

(b) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[1-n]$

(c) **(Challenge)** $h[n] = n\left(\frac{1}{3}\right)^n u[n-1]$

Problem 2: Causality of Systems

(a) Determine if the following systems are causal or non-causal. These systems are not necessarily LTI.

(i) $y[n] = \dfrac{x[n] - x[n-1]}{2}$

(ii) $y[n] = x[-n]$

(iii) $y[n] = \dfrac{1}{2M+1} \displaystyle\sum_{k=-M}^{+M} x[n-k]$

(iv) $y[n+3] - y[n+1] = x[n+2] + x[n]$

Problem 3: Thermostat State Space

Consider a thermostat that is governed by the following **causal** LCCDE:

$$y[n] = k_d(y[n-1] - y[n-2]) + k_p y[n-1] + x[n],$$

where $y[n]$ is the difference between the ambient temperature and the desired temperature (a.k.a error) and $x[n]$ represents heat being added to the system (*e.g*, from appliances, people moving around, *etc*).

The goal of such a system is for the temperature to settle to the desired value (error goes to zero).

This system is inspired by a PD (proportional-derivative) controller, where the output to a control system is set by three terms:

(1) A *proportional* term, $k_p y[n-1]$, which is proportional to the previous error.

(2) A *differential* term, which in discrete-time is a difference term, $k_d(y[n-1] - y[n-2])$.

(3) An *external input* term, which is $x[n]$

Let $x[n] = \delta[n]$ (for instance, in the context of this problem, someone turns on a space heater and promptly turns it off). Assume $y[n] = 0$ for all $n < 0$ (the system starts at rest).

(a) We'll first discover how proportional feedback alone affects the system. For this part only, set $k_d = 0$. Plot the impulse response of the system, $h[n]$ for $k_p = \frac{1}{2}$. Discuss what would happen if $k_p = -\frac{1}{2}$.

(b) Now let's understand how the differential feedback alone would affect the system. For this part only, set $k_p = 0$. Plot the impulse response of the system, $h[n]$, for $k_d = 1/2$, from $n = 0$ to $n = 6$.

(c) Draw a delay-adder-gain block diagram for this LCCDE using the minimum number of delay blocks.

(d) Define *state variables* $q_1[n]$ and $q_2[n]$ to be the output of each delay block.

Label $q_1[n]$, $q_2[n]$, $q_1[n+1]$, and $q_2[n+1]$ on the DAG block diagram from the previous part.

Determine the state-transition equation,

$$\boldsymbol{q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \boldsymbol{B}x[n],$$

and the output equation

$$y[n] = \boldsymbol{C}\boldsymbol{q}[n] + Dx[n],$$

by finding the matrix $\boldsymbol{A}$, vectors $\boldsymbol{B}$ and $\boldsymbol{C}$, and scalar D that complete the above equations.

(e) What is $y[n]$ in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, D, the Kronecker delta, $\delta[n]$, and the unit step function, $u[n]$? Recall that we have set $x[n] = \delta[n]$.

(f) Assume the modes (eigenvalue-eigenvector pairs) of the system are represented by $(\lambda_1, \boldsymbol{v}_1)$ and $(\lambda_2, \boldsymbol{v}_2)$. In terms of the eigenvalues λ_1 and λ_2, when is the system BIBO stable? For this question, assume all modes are observable, i.e. BIBO stability and internally stability are directly correlated.

Hint: Consider applying the eigendecomposition of $A(n)$ to the expression we have for the impulse response.

(g) Find the eigenvalues of the system for arbitrary k_p and k_d.

Then, set $k_p = k_d = \frac{1}{2}$. Is the system internally stable? For these coefficient values, does it function well as a thermostat?