

Problem 1: Feedback Stabilization

Consider the well-structured interconnection H of causal discrete-time LTI systems A and B , as shown in Figure 1. Each individual system is defined on $[\mathbb{Z} \rightarrow \mathbb{C}] \rightarrow [\mathbb{Z} \rightarrow \mathbb{C}]$.

- (a) Determine the composite system's frequency response $H : \mathbb{R} \rightarrow \mathbb{C}$ in terms of the frequency responses A and B of the individual components. Assume the existence of the frequency response.

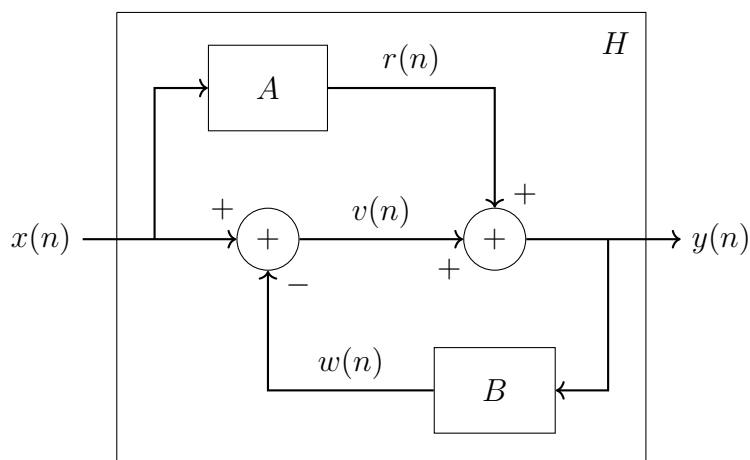


Figure 1: Interconnected DT-LTI system.

For the remaining parts, suppose the systems A and B have impulse response values

$$a[n] = \delta[n - 1] \qquad b[n] = -3\delta[n - 1]$$

(b) Is the system H stable?

(c) Suppose we put H in negative feedback with a constant gain K as shown in fig. 2. For what values of K will the resulting system G be stable? Under this assumption G has a frequency response, so we can use frequency-response arguments (Black's formula).

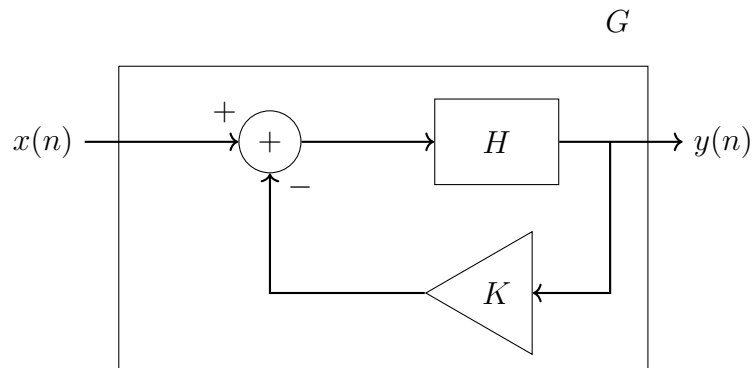


Figure 2: H in negative feedback with a constant gain

Problem 2: DTFS and Change of Basis

In this problem, we will build intuition on Fourier transforms by looking at the DTFS through the lens of change of basis.

We chose the DTFS for this problem because discrete, periodic signals lend themselves well to the vector operations many of you have seen in basic linear algebra. If we look at the space of discrete-time signals with period N , we can represent each signal as a vector $\in \mathbb{R}^N$ consisting of one period of the signal.

- (a) To familiarize yourself with this vector notation, find the fundamental period N of $x[n] = \cos[\pi n] + \sin[\frac{\pi}{2}n]$ and write $x[n]$ as a vector, $\mathbf{x}_v$, in $\mathbb{R}^N$.

- (b) We would like to define an orthogonal basis of complex exponentials for $\mathbb{R}^N$ by using vectors of the form $\mathbf{v}_k[n] = e^{i\omega_k n}$. Remember, we are working in the space of N -periodic functions, so $\mathbf{v}[n]$ must correspond to a complex exponential with period N .

For which values of ω_k does $e^{i\omega_k n}$ have a period of N ? For an arbitrary period N , how many unique vectors of this form exist?

- (c) Show that the set vectors $\{\mathbf{v}_k = e^{i\omega_k n}\}$ using the values of ω_k found in the previous part are orthogonal to each other. A set of vectors is defined as being orthogonal if $\mathbf{v}_l^* \mathbf{v}_m = 0$, $\forall l \neq m$, and the inner product of two complex-valued vectors is defined as follows:

$$\mathbf{a}^* \mathbf{b} = \sum_{n=0}^{N-1} \bar{a}[n] b[n],$$

where $\mathbf{a}^*$ is the conjugate transpose of $\mathbf{a}$ and $\bar{a}[n]$ is the complex conjugate of $a[n]$.

Hint: the formula for the summation of a geometric series is

$$\sum_{k=0}^{N-1} a^k = \frac{a^N - 1}{a - 1}.$$

- (d) What is the norm of $\mathbf{v}_k$? The norm of a vector is the square root of the inner product of that vector with itself.