

Problem 1: Mystery DTFS

Consider a periodic real-valued discrete-time signal x having fundamental period p and an exponential discrete Fourier series expansion

$$x[n] = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} \quad (1)$$

where ω_0 is the fundamental frequency of the signal. The signal has the following properties:

- $x[n + 3] = x[n], \forall n \in \mathbb{Z}$
- $\sum_{n \in \langle p \rangle} x[n] = 0$
- $\sum_{k \in \langle p \rangle} X_k = 0$
- $\sum_{k \in \langle p \rangle} |X_k|^2 = \frac{1}{2}$

Show that the signal x can be expressed as $x[n] = A \cos(Bn + C)$, and determine the parameters A , B , and C .

Problem 2: Proving DTFS Properties

Given a periodic signal x with period p and frequency $\omega_0 = 2\pi/p$, recall the synthesis and analysis equations of DTFS:

$$x[n] = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n}$$

$$X_k = \frac{1}{p} \sum_{n \in \langle p \rangle} x[n] e^{-ik\omega_0 n}$$

Using these equations, prove the following properties:

(a) Conjugate Symmetry: The DTFS coefficients are conjugate symmetric—that is, $X_k = X_{-k}^*$ —if, and only if, x is a real-valued signal.

(b) Evenness: The DTFS coefficients are real and even—that is, $X_k = X_{-k} = X_k^*$ —if, and only if, x is a real-valued and even signal.

Problem 3: What did DTFS know about $H(\omega)$?

Consider the two-point moving average filter $\mathbf{H}$ whose impulse response h and frequency response H are given by:

$$h[n] = \frac{\delta[n] + \delta[n-1]}{2} \text{ and } H(\omega) = \cos(\omega/2)e^{-i\omega/2}$$

Now suppose we construct a discrete-time function g by a periodic replication of h , according to the following: $g[n] = \sum_{l=-\infty}^{\infty} h[n - lN]$, for all integers n and for some integer $N \geq 2$.

(a) Plot $g[n]$ for $N = 4$.

(b) Determine a reasonably simple expression for the coefficient G_k in the discrete-time Fourier series expansion of g . Your expression for G_k must show how the DTFS coefficients relate to the frequency response $H(\omega)$. What do you notice as $N \rightarrow \infty$?

DTFS Properties

The Discrete Time Fourier Series is a method of analyzing periodic discrete time signals. In particular, the idea is that discrete time periodic signals can be expressed as a linear combination of the roots of unity corresponding to the fundamental period of the signal.

Suppose we have a signal $x[n]$ such that $x[n+p] = x[n] \forall n$. Then, x is said to be p periodic. The smallest such p is the fundamental period (denoted as p_x , or N), and the fundamental frequency is $\omega_0 = \frac{2\pi}{p_x}$. We choose one period of samples using the notation $\langle p \rangle$ to denote an interval of p integers (for example, $[0, 1, 2, \dots, p-1]$).

Key Concept: Synthesis Equation: The synthesis equation says that we can decompose x into a linear combination of the N th roots of unity. Specifically,

$$x[n] = \sum_{k \in \langle p_x \rangle} X_k e^{ik\omega_0 n}$$

The DTFS coefficients X_k can be calculated through the **Analysis Equation**:

$$X_k = \frac{1}{p_x} \sum_{n \in \langle p_x \rangle} x[n] e^{-ik\omega_0 n}$$

A natural consequence of the analysis equation is that the DTFS coefficients are also p -periodic; that is, $X_k = X_{k+p}$.

Properties of the DTFS:

1. Conjugate Symmetry: $x[n]$ is a real valued signal if and only if $X_k = X_{-k}^*$
2. Evenness: $x[n]$ is real valued and even ($x[n] = x[-n]$) if and only if $X_k = X_{-k} = X_k^*$, aka the DTFS coefficients are also real valued and even.
3. Oddness: $x[n]$ is real valued and odd ($x[n] = -x[-n]$) if and only if $X_k = -X_{-k} = -X_k^*$, aka the DTFS coefficients are also imaginary valued and odd.
4. Poisson's identity: $x[n] = \sum_{l=-\infty}^{\infty} \delta[n-lp] = \frac{1}{p} \sum_{k=0}^{p-1} e^{ik\omega_0 n}$ where $\omega_0 = \frac{2\pi}{p}$. This says that if x is an impulse train in time, then $X_k = \frac{1}{p}$.
5. Linearity: $\alpha x[n] \iff \alpha X_k$, and $x[n] + y[n] \iff X_k + Y_k$
6. Parseval's Theorem: $\frac{1}{N} \sum_{\langle N \rangle} |x[n]|^2 = \sum_{k=0}^{N-1} |X_k|^2$