Problem 1: Got Filters?

Your friend Arif, who dabbles in stock and foreign-exchange trading on the international financial markets, wants to tell you about a simplified version of one of the tools they use in the "technical analysis" of price data.

Let $x[n]$ be the closing price of stock X on day n. Arif tells you about filters that they use to determine price trends. In particular, he introduces you to a filter that takes the stock price data signal x and produces another signal y according to the following equation:

$$\forall n \in \mathbb{Z}, \quad y[n] = \frac{1}{3}\big[x[n] + x[n-1] + x[n-2]\big].$$

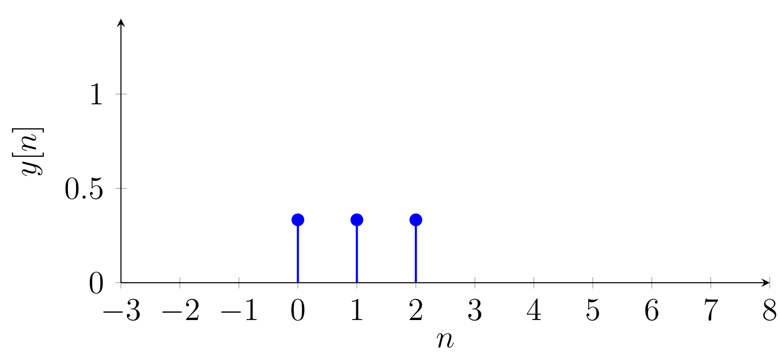
(a) Explain, in plain English, what Arif's toy filter does to the input stock-price signal x.

This filter takes a three-point moving average of the stock price.

(b) Determine the impulse response of the filter, and provide a well-labeled plot of it. That is, determine and plot the response of the filter corresponding to the input signal given by

$$x[n] = \delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$$

$$y[n] = \frac{\delta[n] + \delta[n-1] + \delta[n-2]}{3} = \begin{cases} 1/3, & n = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$



Plot of $y[n]$

(c) Determine, and provide well-labeled plots of, the filter's responses y_1, y_2, and y_3 corresponding to the following hypothetical signals (which do not necessarily represent stock price data):

$$x_1[n] = 1 \text{ for all } n \in \mathbb{Z}$$

$$x_2[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} \quad \text{(Unit-Step Signal)}$$

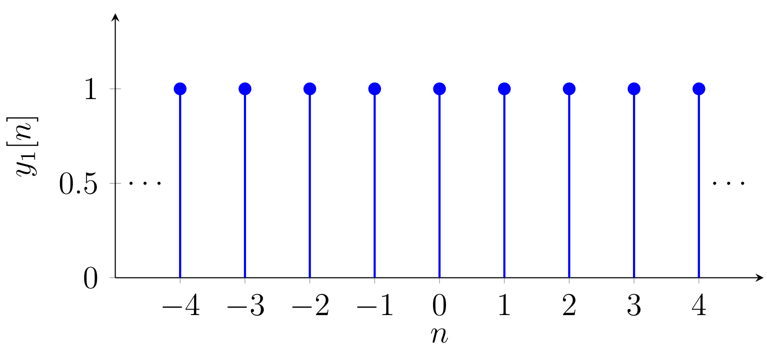
$$x_3[n] = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Based on what you discovered about y_2, explain why Arif would call the filter's output a *lagging indicator* of the stock price's movement.

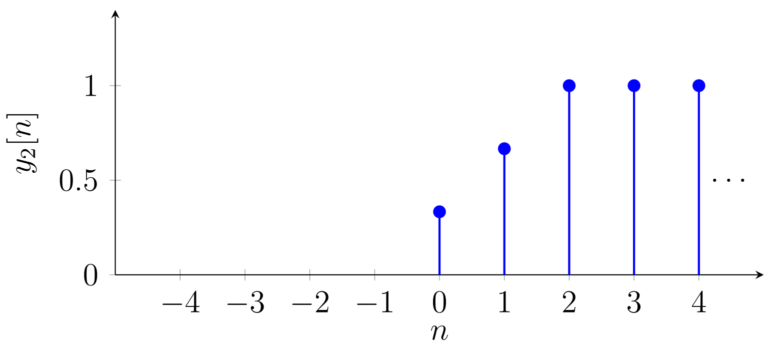
$$y_1[n] = \frac{x_1[n] + x_1[n-1] + x_1[n-2]}{3} = 1$$

$$y_2[n] = \frac{x_2[n] + x_2[n-1] + x_2[n-2]}{3} = \begin{cases} 0, & n < 0 \\ \frac{n+1}{3}, & 0 \leq n < 3 \\ 1, & n \geq 3 \end{cases}$$

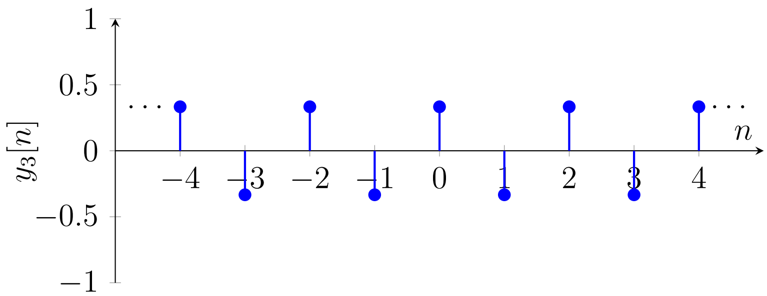
$$y_3[n] = \frac{x_3[n] + x_3[n-1] + x_3[n-2]}{3} = \begin{cases} \frac{1}{3}, & \text{if } n \text{ is even} \\ -\frac{1}{3}, & \text{if } n \text{ is odd} \end{cases}$$



Plot of $y_1[n]$



Plot of $y_2[n]$



Plot of $y_3[n]$

Problem 2:

This problem explores some consequences of Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(a) $\cos\,\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$
$\cos\,\theta = \cos\,\theta + i\frac{1}{2}\sin\,\theta - i\frac{1}{2}\sin\,\theta = \frac{\cos\,\theta + i\,\sin\,\theta}{2} + \frac{\cos\,\theta + i\,\sin\,-\theta}{2} = \frac{e^{i\theta}+e^{-i\theta}}{2}$

(b) $\sin\,\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$
$\sin\,\theta = \sin\,\theta + \frac{1}{2i}\cos\,\theta - \frac{1}{2i}\cos\,\theta = \frac{\cos\,\theta + i\,\sin\,\theta}{2i} + \frac{-\cos\,\theta + i\,\sin\,\theta}{2i} = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

(c) Derive De Moivre's Theorem, where for all integers n,

$$(\cos\theta + i\,\sin\theta)^n = \cos(n\,\theta) + i\,\sin(n\,\theta).$$

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)$$

(d) Using Euler's Formula with a strategically chosen value of θ, derive what American mathematician Benjamin Peirce referred to as the "mysterious formula" [1],

$$\pi = \frac{2}{i}\ln(i).$$

$$e^{i\left(\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

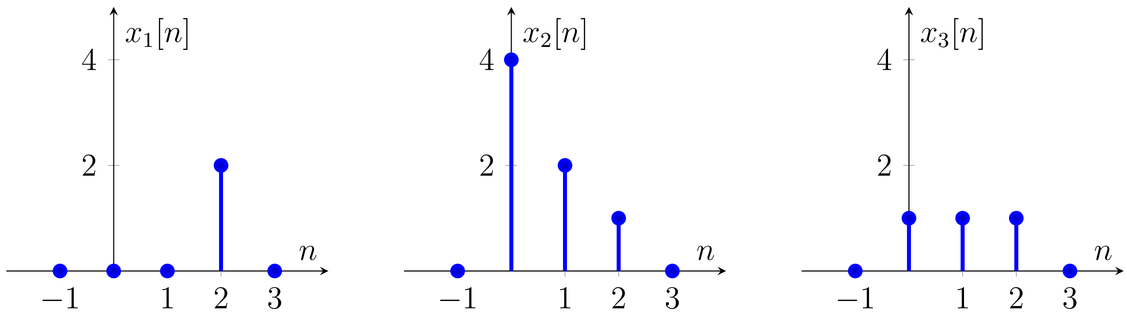
$$\ln\left(e^{i\left(\frac{\pi}{2}\right)}\right) = i\left(\frac{\pi}{2}\right) = \ln(i)$$

$$\pi = \frac{2}{i}\ln(i)$$

[1] For more on this, read the wonderful book by Paul J. Nahin, *An Imaginary Tale: The Story of* $\sqrt{-1}$, Princeton University Press, 1998, ISBN: 0-691-12798-0

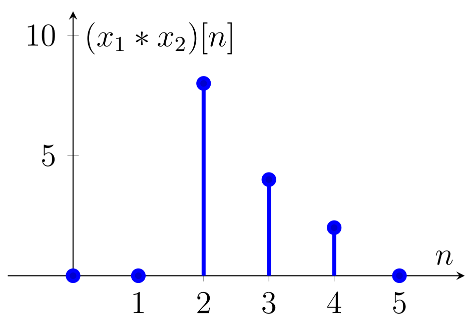
Problem 3: DT Convolution (Practice)

Consider the three discrete-time signals below.

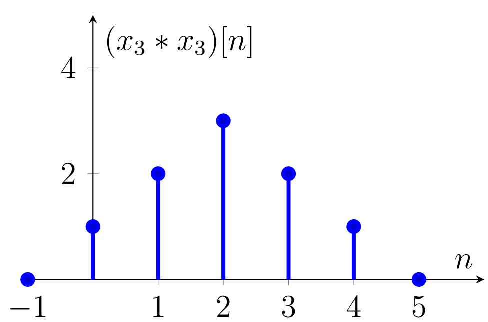


Carefully sketch and label plots of the following signals.

(a) $(x_1 * x_2)[n]$



(b) $(x_3 * x_3)[n]$



Problem 4: Is it LTI?

Each of the following equations describes a relationship that holds between an input signal x and an output signal y of an associated discrete-time system, for all integers n. In each case, determine if the system is (i) linear and (ii) time-invariant.

Here are a few refreshers on definitions. Suppose a system maps a signal x to another signal y.

- The system is said to be *linear* if it satisfies two properties.
  1. Scaling: For any scalar $a \in \mathbb{C}$, ax maps to ay.
  2. Additivity: If x_1 maps to y_1 and x_2 maps to y_2, then $x_1 + x_2$ maps to $y_1 + y_2$, for any signals x_1, x_2.

  One of the easiest ways to show a system is *not* linear is by showing it breaks the scaling property in the case of $a = 0$: a linear system *always* maps an all zero input signal to an all zero output signal. Conversely, if you have a hunch that a system *is* linear, it's often easier to check both properties at once by verifying that $ax_1 + bx_2$ maps to $ay_1 + by_2$, where y_1 is the system's output for x_1 and y_2 is the system's output for x_2.

- The system is said to be *time-invariant* if $x[n - n_d]$ maps to $y[n - n_d]$ for any sample delay $n_d \in \mathbb{Z}$. That is, delaying the input to a time-invariant system will cause it to produce the same output, just delayed by that amount.

(a) $y[n] = ax[n - n_0] + b$, where $a, b \in \mathbb{C}, n_0 \in \mathbb{Z}$. In cases where you answer no: does your answer become yes change if we fix $b = 0$? What if we fix $n_0 = 0$?

(i) **Linear: No.** If we set $x[n] = 0$, or in other words the function x is 0 for all values of n, the system outputs $y[n] = b$, violating the scaling property. If we fix $b = 0$, then yes, the system is linear. The value of n_0 (i.e., the delay factor) does not affect linearity here.

(ii) **Time-invariant: Yes.** The system maps $x[n]$ to $y[n] = ax[n - n_0] + b$. Suppose we delay the input signal x by N samples, where $N \in \mathbb{Z}$, to obtain $x[n - N]$. Then the output is $ax[(n - N) - n_0] = ax[n - N - n_0] = y[n - n_d]$. Delaying any input signal x by n_d samples delays the output by n_d samples, thus the system is time-invariant.

The system here is an *affine* system: a linear system plus a constant offset. Unless the offset is zero, it's not a linear system, as a zero input must always map to a zero output.

(b) $y[n] = x[n] - x[n-1]$

(i) **Linear: Yes.** Scaling the output by some factor α gives the following result: $\alpha y[n] = \alpha(x[n] - x[n-1]) = \alpha x[n] - \alpha x[n-1]$. Scaling the inputs will give the same result. To verify additivity, we can compare the result of combining a set of two outputs: $y_1[n]+y_2[n] = x_1[n]-x_1[n-1]+x_2[n]-x_2[n-1]$ to the system output of a combination of two inputs: $x_1[n]+x_2[n]-(x_1[n-1]+x_2[n-1]) = x_1[n] - x_1[n-1] + x_2[n] - x_2[n-1]$.

(ii) **Time-invariant: Yes.** To verify, we can compare the result of delaying the output $y[n]$ by some arbitrary N to the result of delaying the input $x[n]-x[n-1]$ by some arbitrary N. Delaying the output: $y[n-N] = x[n-N]-x[n-N-1]$ and delaying the inputs: $x[n-N] - x[n-1-N]$ gives equivalent answers.

(c) $y[n] = \ln(x[n])$

(i) **Linear: No.** If we set $x[n] = 0$, the system output is not well-defined. In fact, drawing $\ln(x)$ we can immediately see that $\forall x \leq 0$, $\ln(x)$ is undefined, and by intuition $\ln(x)$ is in and of itself a non-linear function.

(ii) **Time-invariant: Yes.** To verify, we can compare the result of delaying the output $y[n]$ by some arbitrary N to the result of delaying the input $x[n]$ by some arbitrary N. Delaying the output: $y[n-N] = \ln(x[n-N])$ and delaying the inputs: $\ln(x[n-N])$ gives equivalent answers.

(d) $y[n] = x[2n]$

(i) **Linear: Yes.** Let us define some arbitrary input signals $x_1[n]$ and $x_2[n]$, and their corresponding output signals $y_1[n] = x_1[2n]$ and $y_2[n] = x_2[2n]$. Let $x_3[n] = \alpha x_1[n] + \beta x_2[n]$, where $\alpha, \beta \in \mathbb{C}$. Then $y_3[n] = x_3[2n] = \alpha x_1[2n] + \beta x_2[2n] = \alpha y_1[n] + \beta y_2[n]$, which satisfies the scaling and additivity.

(ii) **Time-invariant: No.** Let us define some arbitrary shifted input signal $\hat{x}[n] = x[n - n_d]$, where $n_d \in \mathbb{Z}$. Then the output signal is $\hat{y}[n] = \hat{x}[2n] = x[2n - n_d]$. This is not equivalent to $y[n - n_d] = x[2(n - n_d)] = x[2n - 2n_d]$, so the system is not time-invariant.