Problem 1: DTFT Practice

An LTI system with impulse response $h_1[n] = (1/3)^n u[n]$ is connected in *parallel* with another causal LTI system with impulse response $h_2[n]$. The resulting parallel interconnection has the frequency response

$$\begin{aligned} H(\omega) &= \frac{-12 + 5e^{-i\omega}}{12 - 7e^{-i\omega} + e^{-i2\omega}} \\ &= \frac{-3}{e^{-i\omega} - 3} + \frac{8}{e^{-i\omega} - 4} \end{aligned}$$

(a) Find $h_2[n]$ using a DTFT pair derived in class:

$$\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \alpha e^{-i\omega}} \quad |\alpha| < 1$$

The frequency response of H_1 can be found as

$$h_1[n] = \left(\frac{1}{3}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - (1/3)\, e^{-i\omega}}.$$

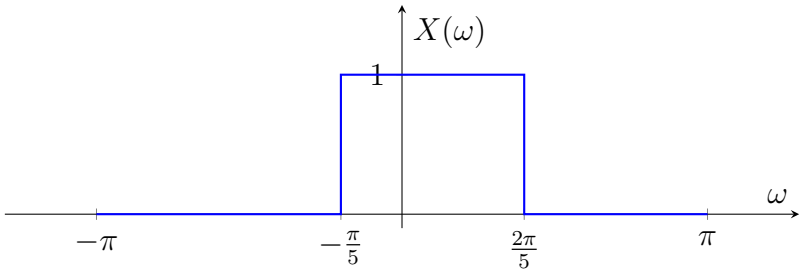
We can do some algebraic manipulation to write $H(\omega)$ as the sum of $H_1(\omega)$ and a second term. As the systems are connected in parallel, $h[n] = h_1[n] + h_2[n]$. By linearity, $H(\omega) = H_1(\omega) + H_2(\omega)$, so this second term is $H_2(\omega)$.

$$H(\omega) = \underbrace{\frac{1}{1 - (1/3)\, e^{-i\omega}}}_{H_1(\omega)} + \underbrace{\frac{8}{(-4)} \frac{1}{1 - (1/4)\, e^{-i\omega}}}_{H_2(\omega)}$$

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n].$$

For the rest of this problem, we will use Fourier transform properties to either compute the DTFT or inverse DTFT of signals.

(b) The spectrum $X(\omega)$ is shown below for $\omega \in [-\pi, \pi]$, and 2π-periodically repeats outside this region. **You do not need to compute any integrals.**



Feel free to use the fact that the inverse DTFT of a spectrum defined as

$$F(\omega) = \begin{cases} 1 & \omega \in [-\omega_c, \omega_c] \\ 0 & \text{otherwise} \end{cases}$$

for $|\omega| < \pi$ (repeating 2π-periodically outside that region) is

$$f[n] = \frac{\sin(\omega_c n)}{\pi n}.$$

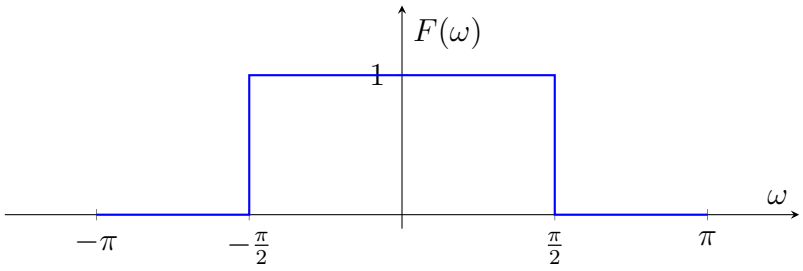
(i) The spectrum is a shifted version of the one given in the hint, so we want to use the **shift in frequency property**:

$$e^{i\omega_c n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_c)$$

(ii) The spectrum is shifted by $\frac{\pi}{10}$ to the right of one symmetric with support on $[-3\pi/10, 3\pi/10]$. So $\omega_c = \frac{\pi}{10}$, and we have:

$$x[n] = e^{i\frac{\pi}{10}n} \frac{\sin(\frac{3\pi}{10}n)}{\pi n}$$

(c) We have some system F whose impulse response $f[n]$'s DTFT $F(\omega)$ is shown below for $\omega \in [-\pi, \pi]$, and 2π-periodically repeats outside this region.:



If the input $x[n] = \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1]$ is passed into this system, what is the corresponding DTFT $Y(\omega)$ of the output?

(i) We can first find the DTFT of the input signal $x[n]$:

$$X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-i\omega n}$$

Plugging in $x[n]$:

$$\begin{aligned} X(\omega) &= \sum_{-\infty}^{\infty} (\frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n+1])e^{-i\omega n} \\ &= \frac{1}{2}e^{-i\omega} + \frac{1}{2}e^{i\omega} \end{aligned}$$

Using inverse Euler's Formula:

$$X(\omega) = \frac{e^{i\omega} + e^{-i\omega}}{2} = cos(\omega)$$

(ii) We can use the Convolution property of the DTFT, which states that:

$$(x * f)[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)F(\omega)$$

And since by definition, we know that:

$$y[n] = (x * f)[n]$$

This implies that:

$$Y(\omega) = X(\omega)F(\omega)$$

Therefore allowing us to find that for $\omega \in [-\pi, \pi]$,:

$$Y(\omega) = \begin{cases} \cos(\omega) & -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

And 2π-periodically repeats outside this region

Problem 2: Mechanical CTFT

For each of the $x(t)$ below, find $X(\omega)$.

(a) $x(t) = e^{i\omega_0 t}$.

This answer is very non-trivial. Let us start with a naive approach and watch it fail.

Naive Approach:

We can try to naively compute the Fourier transform of $x(t)$:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t}dt \\ &= \int_{-\infty}^{\infty} e^{-i\alpha t}dt \end{aligned}$$

where $\alpha = \omega - \omega_0$. We proceed with integration:

$$\begin{aligned} X(\omega) &= \lim_{T\to\infty} \int_{-T}^{T} e^{-i\alpha t}dt \\ X(\omega) &= \lim_{T\to\infty} \frac{1}{-i\alpha}(e^{-i\alpha T} - e^{i\alpha T}) \\ X(\omega) &= \lim_{T\to\infty} \frac{2}{\alpha}\frac{e^{i\alpha T} - e^{-i\alpha T}}{2i} \\ X(\omega) &= \lim_{T\to\infty} \frac{2}{\alpha}\sin(\alpha T) \end{aligned}$$

This final expression is undefined as $\sin(\alpha T)$ as $T \to \infty$ will oscillate. Simplifying the expression above is well outside the scope of this class.

Inverse Approach:

Instead of computing $X(\omega)$, let us instead rewrite $x(t)$ in terms of $X(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$$
$$e^{i\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$$

Looking at the equation above, we can see attempt to use the sifting property! Recall that

$$\int_{-\infty}^{\infty} \delta(t-T)f(t)dt = f(T).$$

So, we let $X(\omega) = 2\pi\delta(\omega - \omega_0)$, we get:

$$e^{i\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$$
$$e^{i\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (2\pi\delta(\omega - \omega_0))e^{i\omega t}d\omega$$
$$e^{i\omega_0 t} = \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{i\omega t}d\omega$$
$$e^{i\omega_0 t} = e^{i\omega_0 t} \text{ by sifting property.}$$

(b) $x(t) = \cos(\omega_0 t + \phi)$.

We can write $x(t) = \frac{e^{i\phi}}{2}e^{i\omega_0 t} + \frac{e^{-i\phi}}{2}e^{-i\omega_0 t}$. Then, using linearity of the Fourier transform, we have

$$F\{x(t)\} = \frac{e^{i\phi}}{2}F\{e^{i\omega_0 t}\} + \frac{e^{-i\phi}}{2}F\{e^{-i\omega_0 t}\}$$

We know $F\{e^{i\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$ and $F\{e^{-i\omega_0 t}\} = 2\pi\delta(\omega + \omega_0)$ from the previous part. So, our final answer is

$$X(\omega) = \pi\left(e^{i\phi}\delta(\omega - \omega_0) + e^{-i\phi}\delta(\omega + \omega_0)\right)$$

(c) $x(t) = Y(t)$ where $Y(\omega)$ is the Fourier transform of some signal $y(t)$.

We know that

$$Y(\omega) = \int_{-\infty}^{\infty} y(\tau) e^{-i\omega\tau} d\tau$$
$$\implies Y(t) = \int_{-\infty}^{\infty} y(\tau) e^{-it\tau} d\tau$$

What we want to find is

$$X(\omega) = \int_{-\infty}^{\infty} Y(t) e^{-i\omega t} dt$$

Since Y is the Fourier transform of y, we can write

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\alpha) e^{i\alpha t} d\alpha$$

Note that we used α instead of ω. We can always change the name of a variable. The expression above looks very similar to $X(\omega)$. If we consider $2\pi y(-\omega)$, we get

$$2\pi y(-\omega) = \int_{-\infty}^{\infty} Y(\alpha) e^{-i\alpha\omega} d\alpha$$

This expression is **exactly** equal to $X(\omega)$. So, we have that

$$X(\omega) = 2\pi y(-\omega).$$

Another way to write this is that

$$F\{Y(t)\} = 2\pi y(-w)$$

(d) $x(t) = R_T(t)$ where $R_T(t)$ is a length T rect:

$$R_T(t) = \begin{cases} 1 & |t| < T \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} R_T(t) e^{-i\omega t} dt \\ &= \int_{-T}^{T} e^{-i\omega t} dt \\ &= \frac{e^{i\omega T} - e^{-i\omega T}}{i\omega} \\ &= \frac{2}{\omega} \frac{e^{i\omega T} - e^{-i\omega T}}{2j} \\ \implies X(\omega) &= \frac{2\sin(\omega T)}{\omega} \end{aligned}$$

(e) $x(t) = \frac{\sin(Tt)}{\pi t}$.

From the previous part, we found that $F\{R_T(t)\} = F_R(\omega) = \frac{2\sin(\omega T)}{\omega}$. Observe that $x(t) = \frac{1}{2\pi} F_R(t)$. The Fourier transform of $x(t)$ is then given by

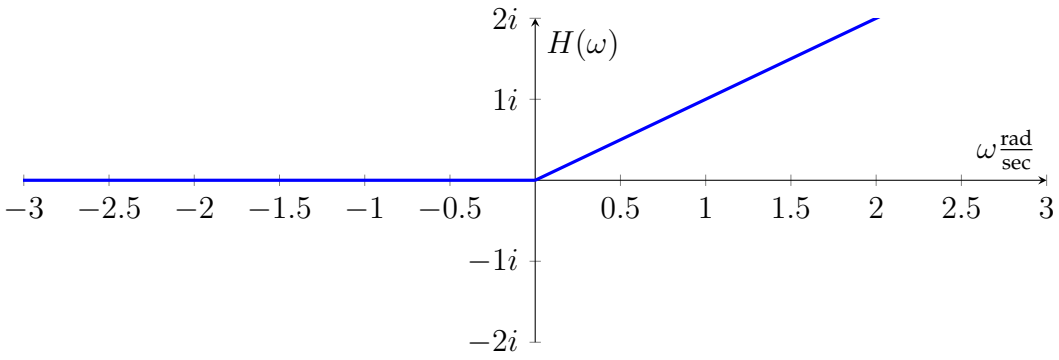
$$X(\omega) = F\{x(t)\} = \frac{1}{2\pi} F\{F_R(t)\}$$

From part (c), we found that $F\{F_R(t)\} = 2\pi R_T(-\omega)$. So, our final result is that

$$X(\omega) = R_T(-\omega) = R_T(\omega)$$

The final equality comes from the fact that $R_T(t)$ is even.

Problem 3: Derivative



$H(\omega)$ is the frequency response of an LTI system and $x(t)$, $y(t)$ is an input-output pair for the system, where

$$x(t) = \begin{cases} e^{i\omega_0 t}\frac{\sin(\omega_0 t)}{\pi t} & t \neq 0 \\ \frac{\omega_0}{\pi} & t = 0 \end{cases}$$

Find $y(t)$ when $t \neq 0$.

Hint: Write $Y(\omega)$ in terms of $X(\omega)$.

We start by finding $X(\omega)$:

$$X(\omega) = R(\omega - \omega_0)$$

Where

$$R(\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & \text{else} \end{cases}$$

Notice that $X(\omega) = 0$ for all $\omega < 0$. So, we can write that

$$Y(\omega) = X(\omega)H(\omega) = i\omega X(\omega)$$

From the derivative property of Fourier transforms, we know that

$$F\left\{\frac{d}{dt}x(t)\right\} = i\omega X(\omega)$$

So, $y(t)$ is just the derivative of $x(t)$. We find this for $t \neq 0$ using chain/product rule to be

$$y(t) = \frac{1}{\pi t^2} e^{i\omega_0 t}\Big((i\omega_0 t - 1)\sin(\omega_0 t) + \omega_0 t\cos(\omega_0 t)\Big) \qquad \forall t \neq 0$$