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Department of EECS

Problem 1: Sinc and Geometry

We know that if we have $x(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$, the CTFT of $x(t)$ is given by

$$X(\omega) = \begin{cases} 1 & |\omega| \leq \pi \\ 0 & |\omega| > \pi \end{cases}.$$

Suppose that we are instead interested in $y(t) = x^2(t) = \text{sinc}^2(t)$. Find and plot $Y(\omega)$.

First we notice that

$$Y(\omega) = \frac{1}{2\pi} X(\omega) * X(\omega)$$

Computing $X(\omega) * X(\omega)$ is easiest to do graphically. We use the formula

$$Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\alpha) X(\omega - \alpha) d\alpha$$

Since $X(\omega)$ is even, we can replace $X(\omega - \alpha)$ with $X(\alpha - \omega)$. Furthermore, $X(\alpha)$ is zero when $|\alpha| > \pi$. So, we have

$$Y(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\alpha) X(\alpha - \omega) d\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\alpha - \omega) d\alpha$$

By graphing out $X(\alpha - \omega)$ in the interval $\alpha \in [-\pi, \pi]$, we see that for

$$\omega \in [-2\pi, 0] \implies Y(\omega) = 1 + \frac{\omega}{2\pi}.$$

Similarly,

$$\omega \in [0, 2\pi] \implies Y(\omega) = 1 - \frac{\omega}{2\pi}.$$

For any other value of $\omega \in (-\infty, -2\pi) \cup (2\pi, \infty)$, we have that $Y(\omega) = 0$.

So, we conclude that

$$Y(\omega) = \begin{cases} 1 - |\frac{w}{2\pi}| & |\omega| < 2\pi \\ 0 & \text{else} \end{cases}$$

Problem 2: A Very Normal Question

The Central Limit Theorem (CLT) states that $\sqrt{n}$ times the average of n independent and identically distributed (i.i.d.) zero-mean, unit-variance random variables converges to a standard Normal distribution as $n \to \infty$. In this question, we frame the CLT in the context of signals, using the fact that summing random variables corresponds to convolving their probability density functions (PDFs). We will prove the CLT using characteristic functions (the CTFT of a distribution's PDF) by showing that the infinite convolution of certain signals converges to a Gaussian.

Consider a non-negative, real-valued signal x whose spectrum $X = \mathcal{F}\{x\}$ has the following properties:

- $X(0) = 1$
- $X'(0) = 0$
- $X''(0) = -1$

(a) For some positive integer n, let $y_n(t) = \sqrt{n}\, x(\sqrt{n}t)$. Find Y_n, the CTFT of y_n, in terms of X.

By the time-frequency scaling property of the Fourier transform, $Y_n(\omega) = \sqrt{n}\frac{1}{\sqrt{n}} X\left(\frac{\omega}{\sqrt{n}}\right) = X\left(\frac{\omega}{\sqrt{n}}\right)$.

(b) Now let's define the signal z_n to be n copies of y_n convolved together, i.e.

$$z_n(t) = \underbrace{(y_n * y_n * \ldots * y_n)}_{n \text{ times}}(t).$$

In the context of probability theory, this is the equivalent to taking $\sqrt{n}$ times the average of n i.i.d. random variables with PDF $x(t)$. So, by the CLT, $\lim_{n\to\infty} z_n(t)$ should be the PDF of the standard Normal distribution.

Determine a reasonably simple expression for Z_n, the CTFT of z_n.

Since convolution in the time domain corresponds to multiplication in the frequency domain, $\mathcal{F}\{z_n\}(\omega) = \mathcal{F}\{y_n\}(\omega)^n = X\left(\frac{\omega}{\sqrt{n}}\right)^n$.

(c) Let's explore what happens as we take more copies of y_n convolved together. Show that $\lim_{n\to\infty} Z_n(\omega) = e^{-\frac{\omega^2}{2}}$.

Hint: Consider evaluating $\lim_{n\to\infty} \ln Z_n(\omega)$ *instead. Alternatively, consider expanding* Y_n *into its second order Taylor expansion. You may find the following limit useful:* $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

Consider $\ln Z_n(\omega) = \ln X\left(\frac{\omega}{\sqrt{n}}\right)^n = n \ln X\left(\frac{\omega}{\sqrt{n}}\right)$. Then, change variables to $s = 1/\sqrt{n}$ and note that $s \to 0$ as $n \to \infty$.

Therefore, we may write $\lim_{n\to\infty} \ln Z_n(\omega) = \lim_{s\to 0} \frac{\ln X(s\omega)}{s^2}$.

Applying L'Hôpital's rule twice, and using the given information about X:

$$\lim_{s\to 0} \frac{\ln X(s\omega)}{s^2} = \lim_{s\to 0} \frac{X'(s\omega)\omega}{X(s\omega)2s} = \lim_{s\to 0} \frac{X''(s\omega)\omega^2}{X'(s\omega)\omega \cdot 2s + X(s\omega)\cdot 2} = -\frac{\omega^2}{2}$$

Therefore, $\lim_{n\to\infty} Z_n(\omega) = e^{\frac{-\omega^2}{2}}$.

Alternative solution:
Recall the second order Taylor expansion for a function f about 0 is $f(\epsilon) = f(0) + f'(0)\epsilon + \frac{1}{2}f''(0)\epsilon^2 + o(\epsilon^2)$, where $\lim_{\epsilon\to 0} \frac{o(\epsilon^2)}{\epsilon^2} = 0$, i.e. the higher order terms go to 0 faster than ϵ^2. Expand $X\left(\frac{\omega}{\sqrt{n}}\right)$ using the given derivatives of X,

$$X\left(\frac{\omega}{\sqrt{n}}\right) = 1 - \frac{\omega^2}{2n} + o\left(\frac{\omega^2}{n}\right).$$

We can neglect the the higher order terms since $\frac{\omega}{\sqrt{n}} \to 0$ as $n \to \infty$, so they decay faster than the initial terms. Then

$$\lim_{n\to\infty} Z_n(\omega) = \lim_{n\to\infty} \left(1 - \frac{\omega^2}{2n} + o\left(\frac{\omega^2}{n}\right)\right)^n = \lim_{n\to\infty} \left(1 - \frac{\omega^2}{2n}\right)^n = e^{-\frac{\omega^2}{2}}$$

(d) Determine an approximate expression for $z_n(t)$ when $n \gg 1$.

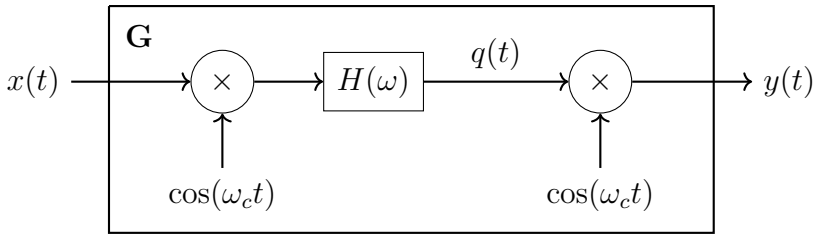
Hint: Recall the Fourier pair: $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\frac{\omega^2}{2}}$.

For large n, the limiting behavior of Z_n is a good approximation. Using the Fourier pair, $z_n(t) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$.

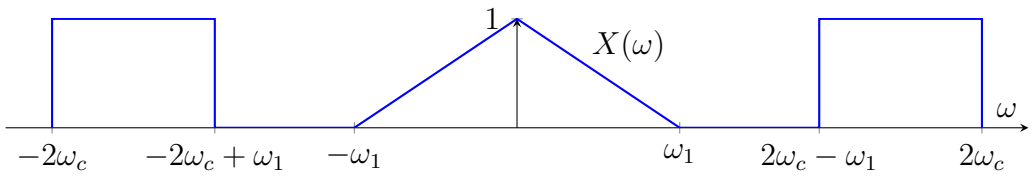
This is the Central Limit Theorem, a fundamental result in probability theory. The conditions that $x(t) \geq 0 \forall t$ and $X(0) = \int_{\infty}^{\infty} x(t)dt = 1$ make x a valid probability density function for a continuous random variable. The other two conditions make a random variable distributed according to x have zero mean and unit variance. Consider n independent copies of the random variable with density x. The density function of the sum of these random variable (scaled by a constant $\frac{1}{\sqrt{n}}$), is given by z_n, i.e. by convolving their scaled densities in the manner above. It turns out (as we have just shown), the probability density function of this new random variable always approaches the probability density function of standard (zero-mean, unit-variance) normally distributed random variable in the limit as $n \to \infty$.

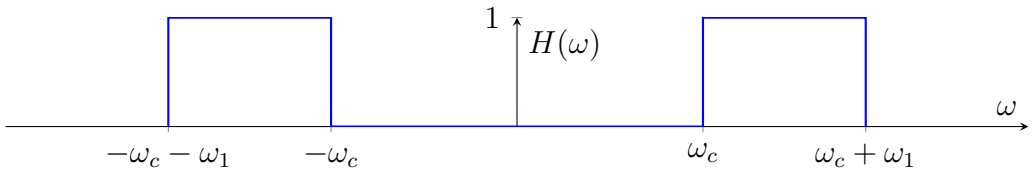
Problem 3: Modulation

A continuous-time system G consists of three processing blocks: modulation, followed by LTI filtering, followed by another modulation, as the figure below indicates.



The spectrum $X(\omega)$ of the input signal x, as well as the frequency response $H(\omega)$ of the LTI filter, are depicted below. Assume that ω_c is sufficiently large; in particular, $\omega_c > 2\omega_1$.





(a) Provide well-labeled plots of $Q(\omega)$ and $Y(\omega)$, the spectra of $q(t)$ and $y(t)$, respectively.

Let $r(t) = x(t)\cos(\omega_c t)$. Then,

$$r(t) = \frac{1}{2}(x(t)e^{-i\omega_c t} + x(t)e^{i\omega_c t})$$

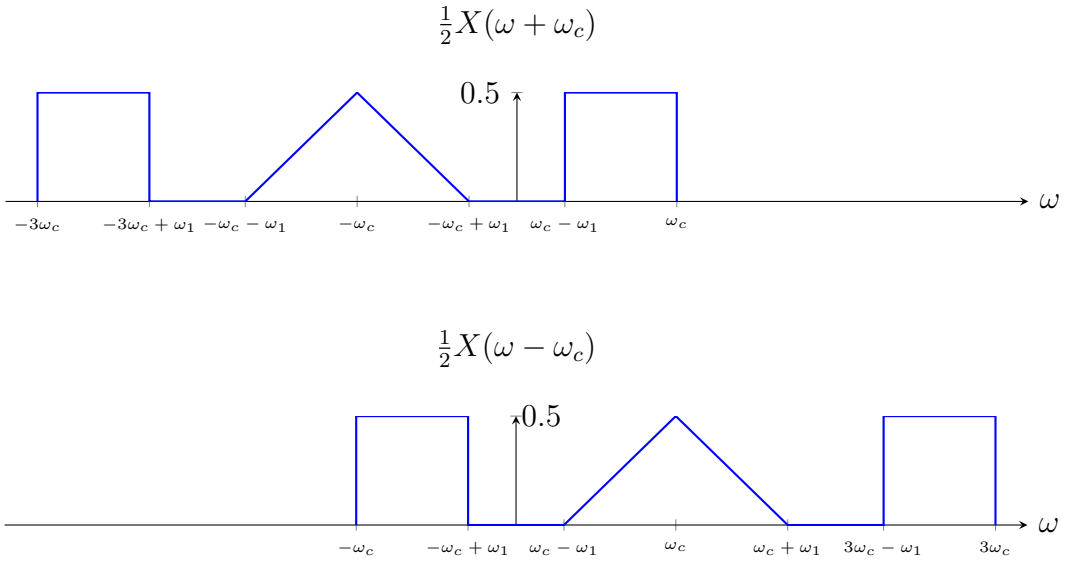
By linearity and the multiplication property of CTFT, we obtain

$$R(\omega) = \frac{1}{2}(X(\omega) * \delta(\omega + \omega_c) + X(\omega) * \delta(\omega - \omega_c))$$

By the sifting property, we obtain

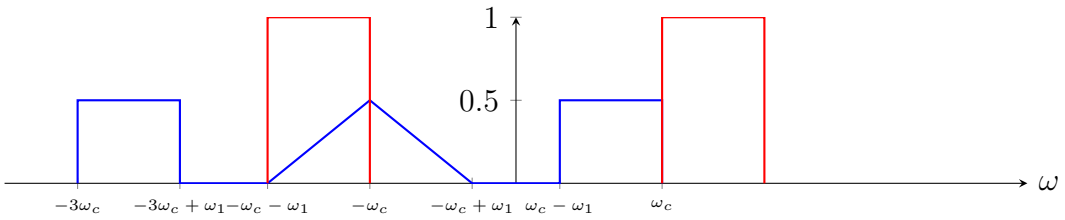
$$R(\omega) = \frac{1}{2}(X(\omega + \omega_c) + X(\omega - \omega_c))$$

Graphically, this becomes:

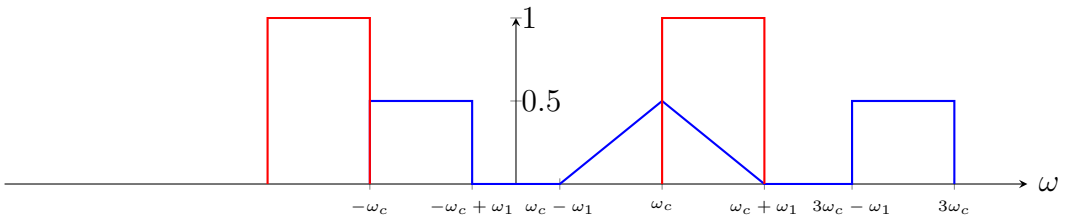


where we draw $\frac{1}{2}X(\omega + \omega_c)$ and $\frac{1}{2}X(\omega - \omega_c)$ separately.
We then apply the filter H, which passes frequencies within the ranges $[-\omega_c - \omega_1, \omega_c]$ and $[\omega_c, \omega_c + \omega_1]$. The passbands are shown in red.

$\frac{1}{2}X(\omega+\omega_c)H(\omega)$

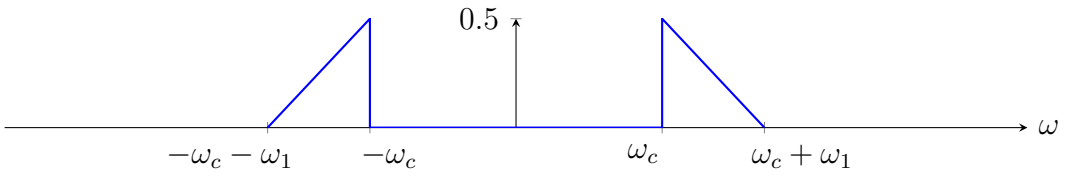


$\frac{1}{2}X(\omega-\omega_c)H(\omega)$



This becomes:

$Q(\omega)$

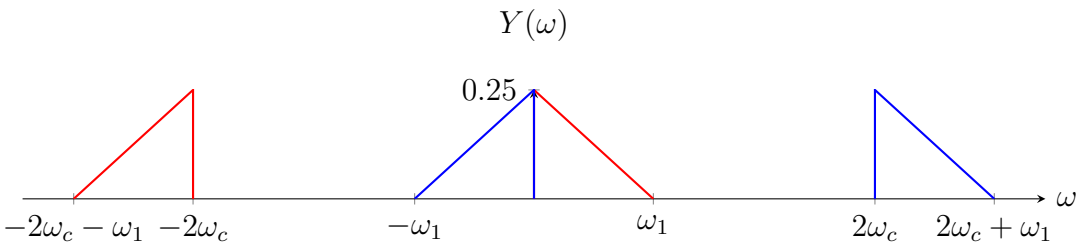


Finally, we modulate by $\cos(\omega_c t)$ again. Just like the first time, we have

$$y(t) = q(t)\cos(\omega_c t) = \frac{1}{2}(q(t)e^{-i\omega_c t} + q(t)e^{i\omega_c t})$$

and

$$Y(\omega) = \frac{1}{2}(Q(\omega+\omega_c) + Q(\omega-\omega_c))$$



Where $\frac{1}{2}Q(\omega + \omega_c)$ is represented in red and $\frac{1}{2}Q(\omega - \omega_c)$ is represented in blue.

(b) Determine a reasonably simply closed-form expression for $h(t)$, the impulse response of the filter.

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega$$

$$H(\omega) = \begin{cases} 1 & |\omega| \in [\omega_c, \omega_c + \omega_1] \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\omega_c-\omega_1}^{-\omega_c} e^{i\omega t}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\omega_c+\omega_1} e^{i\omega t}d\omega \\ &= \frac{1}{2\pi}\left[\frac{1}{it}e^{i\omega t}\Big|_{-\omega_c-\omega_1}^{-\omega_c} + \frac{1}{it}e^{i\omega t}\Big|_{\omega_c}^{\omega_c+\omega_1}\right] \\ &= \frac{1}{\pi t}\left[\frac{1}{2i}\left(e^{-\omega_c t} - e^{-i(\omega_c+\omega_1)t} + e^{i(\omega_c+\omega_1)t} - e^{i\omega_c t}\right)\right] \\ &= \frac{1}{\pi t}\left[\sin((\omega_c+\omega_1)t) - \sin(\omega_c t)\right] \end{aligned}$$