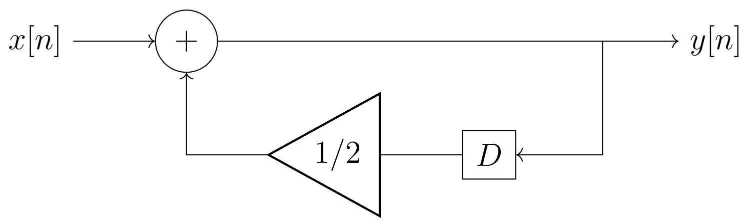
UNIVERSITY OF CALIFORNIA BERKELEY
Department of EECS

Problem 1: Frequency Response from a DAG Block Diagram

Consider a DT LTI system with input $x[n]$ and output $y[n]$ that is implemented by the following block diagram:



(a) Find the LCCDE that describes this system.
The difference equation describing this system is

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

(b) Find the frequency response $H(\omega)$ of this system.
Using the eigenfunction property by setting $x[n] = e^{i\omega n}$ and $y[n] = H(\omega)e^{i\omega n}$,

$$\begin{aligned}
H(\omega)e^{i\omega n} &= e^{i\omega n} + \frac{1}{2}H(\omega)e^{i\omega(n-1)} \\
&= e^{i\omega n} + \frac{1}{2}H(\omega)e^{-i\omega}e^{i\omega n} \\
\left(1 - \frac{1}{2}e^{-i\omega}\right)e^{i\omega n}H(\omega) &= e^{i\omega n} \\
H(\omega) &= \frac{e^{i\omega n}}{\left(1 - \frac{1}{2}e^{-i\omega}\right)e^{i\omega n}} \\
&= \frac{1}{1 - \frac{1}{2}e^{-i\omega}}
\end{aligned}$$

(c) Find the impulse response $h[n]$ of the system. Assume that $h[n] = 0$ for $n < 0$.
Since we're looking for the impulse response, the input is given by $x[n] = \delta[n]$, and we want to find the output $h[n]$.

A good way to start this question is to try writing out the first few terms of the impulse response, and seeing if we can spot a pattern.

Since we know that the output will be zero for $n < 0$, we can start at $n = 0$:

$$x[0] = \delta[0] = 1 \Rightarrow h[0] = \delta[0] + \frac{1}{2}h[-1] = 1 + 0 = 1$$

$$x[1] = \delta[1] = 0 \Rightarrow h[1] = \delta[1] + \frac{1}{2}h[0] = 0 + \frac{1}{2} = \frac{1}{2}$$

$$x[2] = \delta[2] = 0 \Rightarrow h[2] = \delta[2] + \frac{1}{2}h[1] = 0 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

If we keep expanding the pattern, we see that each subsequent term in the impulse response is half of the previous value. Thus, we can write the following expression for the impulse response. Note the unit step $u[n]$, which ensures that the impulse response is zero for $n < 0$.

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

.

Problem 2: Magnitude Response

You are given an LTI system described by the following equation:

$$y(n) = \frac{1}{4}x[n] - \frac{1}{4}x[n-1].$$

(a) For the system given above, find both the impulse response and the frequency response.

Substituting the delta function as our input, we obtain the following for the impulse response:

$$h[n] = \frac{1}{4}\delta[n] - \frac{1}{4}\delta[n-1].$$

We can then find the frequency response by:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-i\omega n} = \frac{1}{4}e^{-i\omega 0} - \frac{1}{4}e^{-i\omega 1} = \frac{1}{4} - \frac{1}{4}e^{-i\omega}.$$

We now have our frequency response, but we can manipulate the expression to make it easier to extract the magnitude and phase response.

The magnitude response can be obtained by factoring out $e^{\frac{-i\omega}{2}}$ to get

$$e^{\frac{-i\omega}{2}}\left(\frac{e^{\frac{i\omega}{2}} - e^{\frac{-i\omega}{2}}}{4}\right).$$

Recognizing that

$$\frac{e^{\frac{i\omega}{2}} - e^{\frac{-i\omega}{2}}}{2i} = \sin\left(\frac{\omega}{2}\right),$$

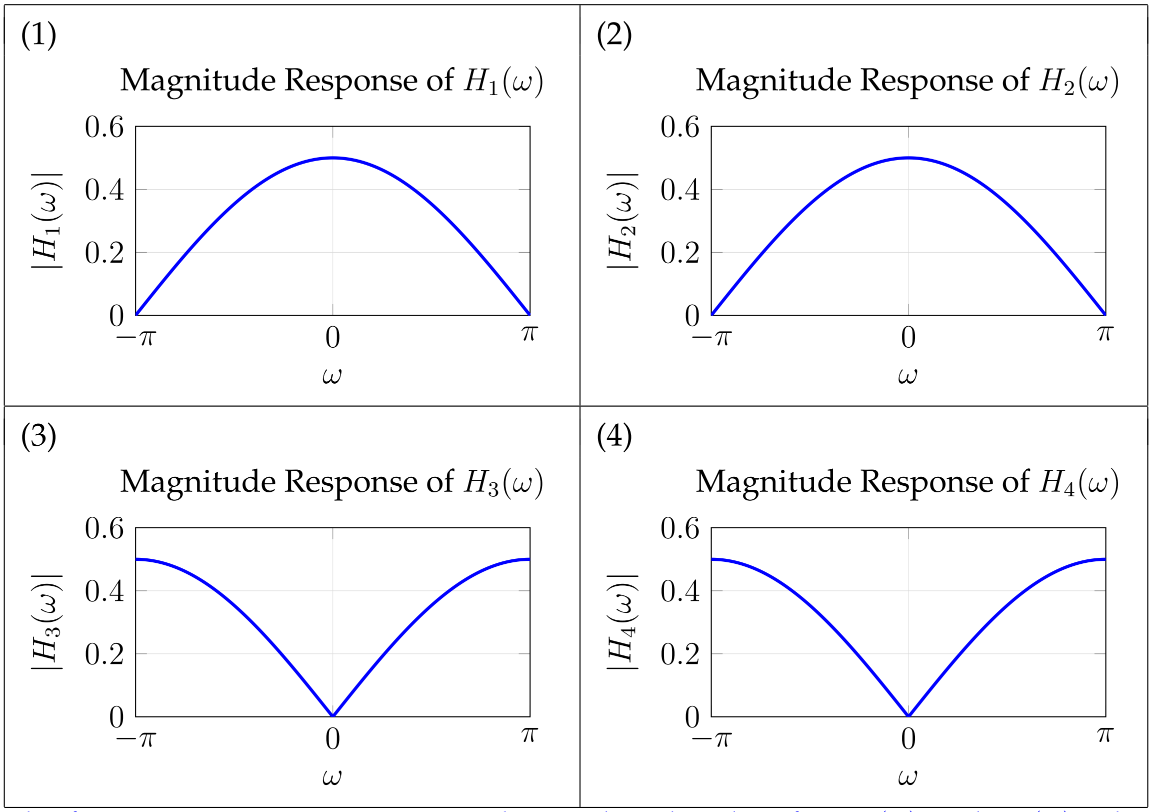
this yields the final solution:

$$H(w) = \frac{i}{2}\sin\left(\frac{\omega}{2}\right)e^{\frac{-i\omega}{2}}.$$

(b) What is the magnitude of the frequency response? Match the frequency response's magnitude to the two correct plots below.

Is this a high-pass or low-pass filter?

$$|H(\omega)| = \left|\frac{i}{2}\right| \left|\sin\left(\frac{\omega}{2}\right)\right| \left|e^{\frac{-i\omega}{2}}\right| = \frac{1}{2}\left|\sin\left(\frac{\omega}{2}\right)\right|.$$



The frequency response's magnitude matches the plots for $H_3(\omega)$ and $H_4(\omega)$. The magnitude response is proportional to $\sin(\frac{\omega}{2})$, so it is 0 when ω is an even multiple of π and reaches a peak when ω is an odd multiple of π.

This is a **high-pass filter** because it attenuates signals with lower frequencies (for discrete-time, low frequencies are close to even multiples of π) and lets signals with higher frequencies (close to odd multiples of π) pass through relatively unchanged.

Problem 3: Making LTI more LIT.

Consider a causal LTI system with frequency response:

$$H(\omega) = \frac{1 + \frac{3}{2}e^{-i\omega}}{1 - \frac{1}{2}e^{-i\omega}}$$

(a) Write a difference equation relating $x[n]$ and $y[n]$ for this system.

$$H(\omega) = \frac{1 + \frac{3}{2}e^{-i\omega}}{1 - \frac{1}{2}e^{-i\omega}}$$

$$H(\omega) - \frac{1}{2}e^{-i\omega}H(\omega) = 1 + \frac{3}{2}e^{-i\omega}$$

$$H(\omega)e^{i\omega n} - \frac{1}{2}e^{-i\omega}H(\omega)e^{i\omega n} = e^{i\omega n} + \frac{3}{2}e^{-i\omega}e^{i\omega n}$$

$$H(\omega)e^{i\omega n} - \frac{1}{2}H(\omega)e^{i\omega(n-1)} = e^{i\omega n} + \frac{3}{2}e^{i\omega(n-1)}$$

We pattern match $x[n] = e^{i\omega n}$ and $y[n] = H(\omega)e^{i\omega n}$ to find our LCCDE:

$$y[n] - \frac{1}{2} \cdot y[n-1] = x[n] + \frac{3}{2} \cdot x[n-1]$$

$$\implies y[n] = x[n] + \frac{3}{2} \cdot x[n-1] + \frac{1}{2} \cdot y[n-1]$$

(b) Draw a block diagram implementing this system with only one delay block.

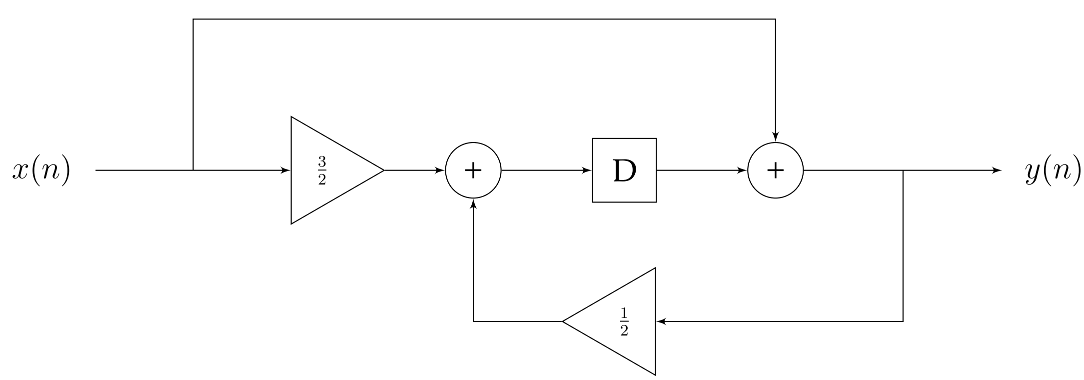


Figure 1: Solution to 3b

(c) Determine the impulse response $h[n]$.

Since the system is causal, we know that $h(n < 0) = 0$. From here it is easiest to build the sequence one element at a time:

$$h[0] = \delta[0] + \frac{3}{2} \cdot \delta[-1] + \frac{1}{2} \cdot h[-1] = 1 + \frac{3}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 1$$

$$h[1] = \delta[1] + \frac{3}{2} \cdot \delta[0] + \frac{1}{2} \cdot h[0] = 0 + \frac{3}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 2$$

$$h[2] = \delta[2] + \frac{3}{2} \cdot \delta[1] + \frac{1}{2} \cdot h[1] = 0 + \frac{3}{2} \cdot 0 + \frac{1}{2} \cdot 2 = 2\left(\frac{1}{2}\right)$$

$$h[3] = \delta[3] + \frac{3}{2} \cdot \delta[2] + \frac{1}{2} \cdot h[2] = 0 + \frac{3}{2} \cdot 0 + \frac{1}{2} \cdot 2(\frac{1}{2}) = 2\left(\frac{1}{2}\right)^2$$

$$\ldots$$

$$h[n] = \delta[n] + 2 \cdot \left(\frac{1}{2}\right)^{n-1} u[n-1]$$