Problem 1: A horse walks into a system, it's a stable stable.

The following are the impulse responses of discrete-time LTI systems. Let $u[n]$ denote the unit step function. Determine whether each system is causal and/or stable.

(a) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[n-1]$

Causal because $h[n] = 0$ for $n < 0$.

Unstable because the second term diverges as $n \to \infty$.

(b) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[1-n]$

We can rewrite $h[n]$ as

$$h[n] = \begin{cases} \left(-\frac{1}{2}\right)^n & \text{if } n \geq 2, \\ (1.01)^n & \text{if } n \leq -1, \\ -\frac{1}{2} + 1.01 & \text{if } n = 1, \\ 2 & \text{if } n = 0. \end{cases}$$

Not causal because $h[n] \neq 0$ for $n < 0$.

Stable because $\sum_n |h[n]| < \infty$.

$$\sum_{n=-\infty}^{\infty} |h[n]| = 2 + 0.51 + \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=-\infty}^{-1} (1.01)^n$$

Recall that for $|r| < 1$, the series $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Applying this formula,

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2}$$
$$\sum_{n=-\infty}^{-1} (1.01)^n = \sum_{m=1}^{\infty} \left(\frac{1}{1.01}\right)^m = \frac{1}{1.01} \cdot \frac{1}{1-\frac{1}{1.01}}$$

From these calculations, we can see that $\sum_n |h[n]|$ is finite, and hence the system is stable.

(c) **(Challenge)** $h[n] = n\left(\frac{1}{3}\right)^n u[n-1]$

Causal because $h[n] = 0$ for $n < 0$.

Stable because $\sum_{n=-\infty}^{\infty} |h[n]| = 0.75 < \infty$. If we write out the summation, we get

$$B = \sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \cdots$$

This looks almost like a geometric series, but not quite, so lets try to massage it into a more familiar form. To get something closer to a geometric series, we want a 1 in the numerator of every term in the sum, so lets rewrite each fraction without a 1 in the numerator as a summation.

$$B = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} + \sum_{n=2}^{\infty} \frac{1}{3^n} + \sum_{n=3}^{\infty} \frac{1}{3^n} + \cdots$$

In the last equality, we have effectively swapped the summations by leveraging the commutative property of addition. Each of these sums is itself an infinite geometric series, so we can apply the infinite geometric series formula used in the last subproblem.

$$B = \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{3^2}}{1-\frac{1}{3}} + \frac{\frac{1}{3^3}}{1-\frac{1}{3}} + \cdots = \frac{1}{1-\frac{1}{3}} \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{3}{2} \cdot \frac{1}{3} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$$

Problem 2: Causality of Systems

(a) Determine if the following systems are causal or non-causal. These systems are not necessarily LTI.

(i) $y[n] = \dfrac{x[n] - x[n-1]}{2}$

(ii) $y[n] = x[-n]$

(iii) $y[n] = \dfrac{1}{2M+1} \displaystyle\sum_{k=-M}^{+M} x[n-k]$

(iv) $y[n+3] - y[n+1] = x[n+2] + x[n]$

(i) Yes.

(ii) No. Note that this system is not LTI.

(iii) Could be causal. If $M \neq 0$, this is an example of a noncausal averaging system. Expanding the sum, note that the output depends on $x[n+M]$. If $M = 0$, the system is $y[n] = x[n]$, which is causal.

(iv) Yes. Equivalent to $y[n] - y[n-2] = x[n-1] + x[n-3]$.

Problem 3: Thermostat State Space

Consider a thermostat that is governed by the following **causal** LCCDE:

$$y[n] = k_d(y[n-1] - y[n-2]) + k_p y[n-1] + x[n],$$

where $y[n]$ is the difference between the ambient temperature and the desired temperature (a.k.a error) and $x[n]$ represents heat being added to the system (*e.g*, from appliances, people moving around, *etc*).

The goal of such a system is for the temperature to settle to the desired value (error goes to zero).

This system is inspired by a PD (proportional-derivative) controller, where the output to a control system is set by three terms:

(1) A *proportional* term, $k_p y[n-1]$, which is proportional to the previous error.

(2) A *differential* term, which in discrete-time is a difference term, $k_d(y[n-1] - y[n-2])$.

(3) An *external input* term, which is $x[n]$

Let $x[n] = \delta[n]$ (for instance, in the context of this problem, someone turns on a space heater and promptly turns it off). Assume $y[n] = 0$ for all $n < 0$ (the system starts at rest).

(a) We'll first discover how proportional feedback alone affects the system. For this part only, set $k_d = 0$. Plot the impulse response of the system, $h[n]$ for $k_p = \frac{1}{2}$. Discuss what would happen if $k_p = -\frac{1}{2}$.

For these coefficient values, the LCCDE is

$$y[n] = \frac{1}{2} y[n-1] + x[n].$$

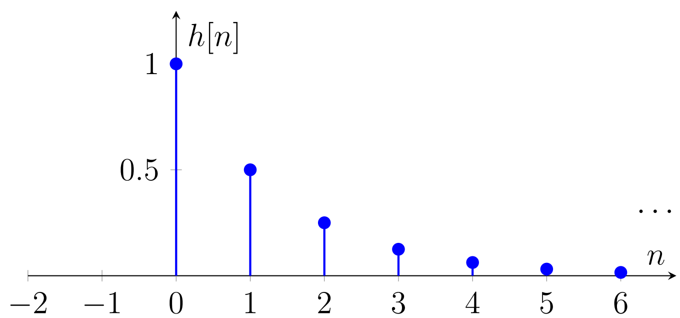
To find the impulse response, plug in $x[n] = \delta[n]$. The corresponding output is the impulse response.

We find the impulse response by examining the response of the system at different time steps, *i.e.*, for different values of n. As the system is causal (like all real life systems are), $h[n] = 0, \forall n < 0$.

So, starting at $n = 0$,

$$
\begin{aligned}
h[0] &= \frac{1}{2}h[-1] + \delta[0] = 1 \\
h[1] &= \frac{1}{2}h[0] + \delta[1] = \frac{1}{2}. \\
h[2] &= \frac{1}{2}h[1] + \delta[2] = \frac{1}{4} \\
&\vdots
\end{aligned}
$$

Notice, overall, $h[n] = \frac{1}{2^n}u[n]$.

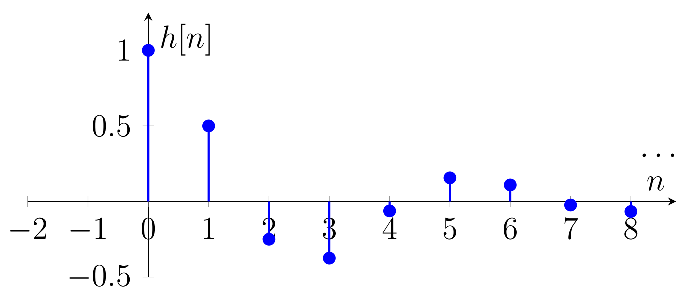


If we had $k_p = -\frac{1}{2}$, the impulse response would alternate between positive and negative values (1, $-\frac{1}{2}$, $\frac{1}{4}$, *etc.*)

(b) Now let's understand how the differential feedback alone would affect the system. For this part only, set $k_p = 0$. Plot the impulse response of the system, $h[n]$, for $k_d = 1/2$, from $n = 0$ to $n = 6$.

Using the same method as above:

$$\begin{aligned}
h[0] &= \frac{1}{2}(h[-1] - h[-2]) + \delta[0] = 1 \\
h[1] &= \frac{1}{2}(h[0] - h[-1]) + \delta[1] = \frac{1}{2}. \\
h[2] &= \frac{1}{2}(h[1] - h[0]) + \delta[2] = -\frac{1}{4} \\
h[3] &= \frac{1}{2}(h[2] - h[1]) + \delta[3] = -\frac{3}{8} \\
h[4] &= \frac{1}{2}(h[3] - h[2]) + \delta[4] = -\frac{1}{16} \\
h[5] &= \frac{1}{2}(h[4] - h[3]) + \delta[5] = \frac{5}{32} \\
h[6] &= \frac{1}{2}(h[5] - h[4]) + \delta[6] = \frac{7}{64} \\
&\vdots
\end{aligned}$$

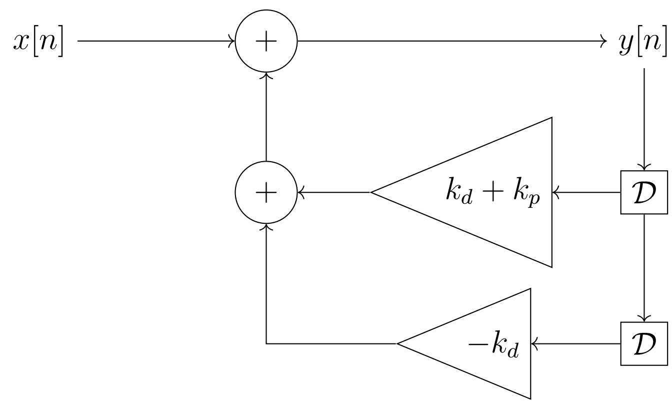


Here we see that the impulse decays with oscillation. As you will see later in class, this graph is composed of *two* exponentially decaying and oscillating terms.

(c) Draw a delay-adder-gain block diagram for this LCCDE using the minimum number of delay blocks.

Re-arranging terms in the LCCDE,

$$y[n] = (k_d + k_p)y[n-1] - k_d y[n-2] + x[n].$$



(d) Define *state variables* $q_1[n]$ and $q_2[n]$ to be the output of each delay block.

Label $q_1[n]$, $q_2[n]$, $q_1[n+1]$, and $q_2[n+1]$ on the DAG block diagram from the previous part.

Determine the state-transition equation,

$$\boldsymbol{q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \boldsymbol{B}x[n],$$

and the output equation

$$y[n] = \boldsymbol{C}\boldsymbol{q}[n] + Dx[n],$$

by finding the matrix $\boldsymbol{A}$, vectors $\boldsymbol{B}$ and $\boldsymbol{C}$, and scalar D that complete the above equations.

$q_1[n]$ is the output of the upper delay block (*i.e.*, $y[n-1]$), and $q_2[n]$ is the output of the lower delay block (*i.e.*, $y[n-2]$).

To find $q[n+1]$ we go forward in time one step (undo the delay block). And so, $q_1[n+1]$ is the *input* of the upper delay block (*i.e.*, $y[n]$) and $q_2[n+1]$ is the input of the lower delay block (*i.e.*, $y[n-1] = q_1[n]$).

For the state-transition equation, we use the DAG block diagram to get an expression for $\boldsymbol{q}[n+1]$ in terms of the state variables and input only:

$$q_1[n+1] = y[n] = (k_d + k_p)q_1[n] - k_d q_2[n] + x[n],$$
$$q_2[n+1] = y[n-1] = q_1[n].$$

Overall, we have

$$\boldsymbol{q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} k_d + k_p & -k_d \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$$
$$\implies \boldsymbol{A} = \begin{bmatrix} k_d + k_p & -k_d \\ 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For the output equation, we also read an expression for $y[n]$ from the DAG block diagram:

$$y[n] = \begin{bmatrix} k_d + k_p & -k_d \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + x[n] \implies \boldsymbol{C} = \begin{bmatrix} k_d + k_p & -k_d \end{bmatrix}, D = 1.$$

(e) What is $y[n]$ in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, D, the Kronecker delta, $\delta[n]$, and the unit step function, $u[n]$? Recall that we have set $x[n] = \delta[n]$.

Since the system is causal and our state vector only depends on $y[n-1]$ and $y[n-2]$, we have $\boldsymbol{q}[n] = 0, \forall n \leq 0$.

Starting with $n = 0$, we write out the state transition equation as follows:

$$\begin{aligned}
\boldsymbol{q}[1] &= \boldsymbol{A}\boldsymbol{q}[0] + \boldsymbol{B}\delta[0] = 0 + \boldsymbol{B} = \boldsymbol{B} \\
\boldsymbol{q}[2] &= \boldsymbol{A}\boldsymbol{q}[1] + \boldsymbol{B}\delta[1] = \boldsymbol{A}\boldsymbol{B} + 0 = \boldsymbol{A}\boldsymbol{B} \\
\boldsymbol{q}[3] &= \boldsymbol{A}\boldsymbol{q}[2] + \boldsymbol{B}\delta[2] = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{B}) + 0 = \boldsymbol{A}^2\boldsymbol{B} \\
&\cdots
\end{aligned}$$

For $n < 1$, $\boldsymbol{q}[n] = 0$, and, for $n \geq 1$, $\boldsymbol{q}[n] = \boldsymbol{A}^{n-1}\boldsymbol{B}$, so

$$\boldsymbol{q}[n] = \boldsymbol{A}^{n-1}\boldsymbol{B}u[n-1].$$

Using the output equation to get $y[n]$,

$$y[n] = \boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{B}u[n-1] + D\delta[n].$$

(f) Assume the modes (eigenvalue-eigenvector pairs) of the system are represented by $(\lambda_1, \boldsymbol{v}_1)$ and $(\lambda_2, \boldsymbol{v}_2)$. In terms of the eigenvalues λ_1 and λ_2, when is the system

BIBO stable? For this question, assume all modes are observable, i.e. BIBO stability and internally stability are directly correlated.

Hint: Consider applying the eigendecomposition of $A(n)$ to the expression we have for the impulse response.

We can make use of the impulse response $h(n)$ result we found last week to solve this problem. We will represent this in terms of the modes of A.

Define the eigendecomposition of A as $V\Lambda V^{-1}$, where

$$V = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}; \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Plugging this eigendecomposition into the equation for $h(n)$ and simplifying,

$$h(n) = CV \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} V^{-1} B u(n-1) + D\delta(n).$$

This system is BIBO stable when the impulse response is absolutely summable. Let us consider three cases for the eigenvalues of A:

(1) $|\lambda_1| > 1$ or $|\lambda_2| > 1$: The λ^{n-1} terms grow exponentially over time, meaning that the magnitude of $h(n)$ also grows exponentially. The system is BIBO unstable.

(2) $|\lambda_1| < 1$ and $|\lambda_2| < 1$: The λ^{n-1} terms decay exponentially over time, as does the magnitude of $h(n)$. The impulse response is therefore absolutely summable, making the system BIBO stable.

(3) One eigenvalue has magnitude 1 and the other has magnitude ≤ 1: One or both of the λ^{n-1} terms remain constant as $n \to \infty$. There is at least one component of $h(n)$ that does not decay to 0, meaning the impulse response is not absolutely summable. The system is not BIBO stable.

(g) Find the eigenvalues of the system for arbitrary k_p and k_d.

Then, set $k_p = k_d = \frac{1}{2}$. Is the system internally stable? For these coefficient values, does it function well as a thermostat?

To find the eigenvalues of system, we solve the equation $\det(\lambda I - A) = 0$ (note: this is equivalent to saying $\det(A - \lambda I) = 0$):

$$0 = \det \begin{bmatrix} \lambda - (k_d - k_p) & k_d \\ -1 & \lambda \end{bmatrix} = \lambda(\lambda - k_d - k_p) + k_d = \lambda^2 - (k_d + k_p)\lambda + k_d.$$

Using the quadratic formula,

$$\lambda = \frac{k_d + k_p \pm \sqrt{(k_d + k_p)^2 - 4k_d}}{2}.$$

Setting $k_p = k_d = \frac{1}{2}$,

$$\lambda = \frac{1 \pm \sqrt{1 - 2}}{2} = \frac{1 \pm i}{2}.$$

The magnitude of both eigenvalues is < 1, so the system is internally stable. Thus the system would work as a thermostat, but we haven't characterized how long it takes for the system to converge to the true temperature. This trait, known as the settling time, is covered more concretely in classes like EECS128.