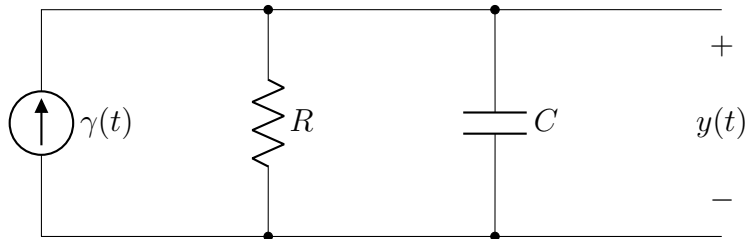


Problem 1: Parallel RC Circuit

Consider the following parallel RC circuit, driven by a current source $\gamma(t)$, with output $y(t)$ the voltage across the parallel combination of R and C .



By KCL at the top node, the source current splits between the capacitor ($C\dot{y}$) and the resistor (y/R), so the system is described by the differential equation

$$C\dot{y}(t) + \frac{1}{R}y(t) = \gamma(t)$$

Note: You don't need to know how to read the circuit to answer this problem.

- (a) Find the impulse response $h(t)$. Assume the form of impulse response $h(t) = Ae^{st}u(t)$.

Let $\gamma(t) = \delta(t)$, so $y(t) = h(t)$:

$$C \frac{d(Ae^{st}u(t))}{dt} + \frac{Ae^{st}u(t)}{R} = \delta(t)$$

Since $\frac{d(Ae^{st}u(t))}{dt} = Ase^{st}u(t) + Ae^{st}\delta(t)$, and $e^{st}\delta(t) = \delta(t)$, we find

$$A \left(Cs + \frac{1}{R} \right) e^{st}u(t) + AC \delta(t) = \delta(t)$$

Matching the $\delta(t)$ terms gives $AC = 1$, so $A = \frac{1}{C}$. Matching the $u(t)$ terms requires $Cs + \frac{1}{R} = 0$, so $s = -\frac{1}{RC}$. Therefore

$$h(t) = \frac{1}{C} e^{-\frac{t}{RC}} u(t)$$

(b) Find the frequency response $H(\omega)$ using the eigenfunction property.

Let $\gamma(t) = e^{j\omega t}$, and $y(t) = H(\omega)e^{j\omega t}$, and plug them into the differential equation

$$\begin{aligned} C \frac{d(H(\omega)e^{j\omega t})}{dt} + \frac{1}{R} H(\omega)e^{j\omega t} &= e^{j\omega t} \\ \left(Cj\omega + \frac{1}{R} \right) H(\omega)e^{j\omega t} &= e^{j\omega t} \\ H(\omega) &= \frac{1}{\frac{1}{R} + j\omega C} = \frac{R}{1 + j\omega RC} \end{aligned}$$

(c) Find the frequency response using the formula $H(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$.

The Frequency Response $H(\omega)$ is

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} \frac{1}{C} e^{-\frac{t}{RC}} u(t) e^{-j\omega t} dt \\ &= \frac{1}{C} \int_0^{\infty} e^{-(\frac{1}{RC} + j\omega)t} dt \\ &= -\frac{1}{C} \frac{1}{\frac{1}{RC} + j\omega} e^{-(\frac{1}{RC} + j\omega)t} \Big|_0^{\infty} \\ &= \frac{1}{C} \frac{1}{\frac{1}{RC} + j\omega} = \frac{R}{1 + j\omega RC} = H(\omega) \end{aligned}$$

The two results are identical: the frequency response is the Fourier transform of the impulse response. (The upper limit vanishes since $\frac{1}{RC} > 0$.)

(d) Calculate and sketch the magnitude response $|H(\omega)|$ and $\angle H(\omega)$ as a function of the frequency ω . Is it a low-pass, high-pass or band-pass filter?

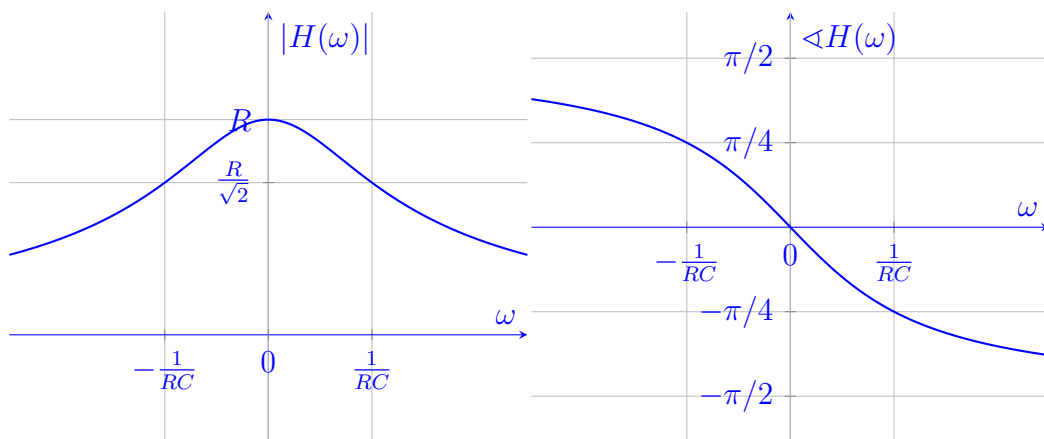
$$|H(\omega)| = \frac{R}{|1 + j\omega RC|} = \frac{R}{\sqrt{1 + (RC\omega)^2}}$$

It is an even function of ω , and $|H(\omega)|$ takes its maximum value R at $\omega = 0$. At $\omega = \pm \frac{1}{RC}$ it falls to $\frac{R}{\sqrt{2}}$.

$$\text{Re}\{H(\omega)\} = \frac{R}{1 + (RC\omega)^2}$$

$$\text{Im}\{H(\omega)\} = \frac{-\omega R^2 C}{1 + (RC\omega)^2}$$

$$\angle H(\omega) = \arctan \frac{\text{Im}\{H(\omega)\}}{\text{Re}\{H(\omega)\}} = -\arctan(\omega RC)$$



The magnitude is largest at $\omega = 0$ and decays toward 0 as $|\omega| \rightarrow \infty$, so it is a low-pass filter.

Problem 2: Causality and BIBO Stability, the Sequel

For each of the following systems, determine whether or not they are causal and/or BIBO stable. Be careful, as they are not necessarily LTI.

We can prove that an LTI system is causal by showing that its impulse response is 0 for all $n < 0$ in discrete time and all $t < 0$ in continuous time. For BIBO stability, we can show that the impulse response is absolutely summable (discrete) or absolutely integrable (continuous).

For non-LTI systems, however, we must use the definitions of causality and BIBO stability to prove either property:

- For causality, show that $y(t_0)$ only depends on $x(t \leq t_0)$, $\forall t_0$. To show that a system is not causal, find a timepoint t_0 such that $y(t_0)$ depends on $x(t_1 > t_0)$. More rigorously, define two input signals $x_1(t)$ and $x_2(t)$ such that $x_1(t) = x_2(t)$ for all $t < t_0$. Prove that the corresponding outputs, $y_1(t)$ and $y_2(t)$, are also equal for all $t < t_0$. To disprove causality, find two inputs $x_1(t)$ and $x_2(t)$ that are equal before time t_0 and show that their outputs differ before time t_0 .
- For BIBO stability, define a bounded input $x(t)$ such that $|x(t)| \leq B_x < \infty$ and show that $|y(t)| \leq B_y < \infty$ for some finite constant B_y . To disprove BIBO stability, find a bounded input such that the output grows unbounded.

Analogous definitions exist for discrete time.

(a) $y(t) = x(-e^{-\alpha t})$, $\alpha \in \{-1, 0, 1\}$.

This system is **causal** if $\alpha = 1$ and **not causal** if $\alpha = 0, -1$.

Let's consider each case separately:

- $\alpha = -1$: $y(t) = x(-e^t)$. Examine $t = -1$: $y(-1) = x(-e^{-1}) = x(-\frac{1}{e})$. $-\frac{1}{e} \approx -0.37 > -1$, so $y(t)$ is "peeking ahead" at $x(-\frac{1}{e})$.
- $\alpha = 0$: $y(t) = x(-e^0) = x(-1)$. For any $t < -1$ (for instance, $t = -2$), $y(t)$ depends on a future value of $x(t)$, so the system is not causal.
- $\alpha = 1$: $y(t) = x(-e^{-t})$. We can show that $-e^{-t} \leq t$, $\forall t$, so $y(t)$ never depends on future values of the input.

For $t \geq 0$, $-e^{-t} < 0 \leq t$. For $t < 0$, $-e^{-t} = -e^{|t|}$, which is $< t$ for all negative t . To convince yourself of this fact, you can plot t and $-e^{|t|}$ and verify that $-e^{|t|}$ is always below t .

This system is **BIBO stable**.

Consider a bounded input $x(t)$ such that $|x(t)| \leq B_x$. We would like to show that the output is also bounded: $|y(t)| \leq B_y$. By the definition of the system, $y(t) = x(-e^{-\alpha t})$. We know that every value of $x(t)$ is bounded by B_x , so $|y(t)| = |x(-e^{-\alpha t})| \leq B_x$, so $y(t)$ is bounded by B_x as well.

(b) $y(t) = \int_{-5|t|}^t x(\tau) d\tau.$

This system is **causal**.

We know that $-5|t| < t, \forall t$ (if t is positive, $-5|t|$ is negative, and if t is negative, $-5|t|$ is a larger negative number), so both bounds of the integral are $\leq t$. This means that $y(t) = \int_{-5|t|}^t x(\tau) d\tau$ never depends on future values of $x(t)$, and the system is therefore causal.

This system is **not BIBO stable**.

Consider the input signal $x(t) = 1$, which is bounded by 1. The output is

$$y(t) = \int_{-5|t|}^t d\tau = \begin{cases} \int_{5t}^t d\tau = -4t, & t \leq 0; \\ \int_{-5t}^t d\tau = 6t, & t > 0, \end{cases}$$

which grows unbounded as t approaches ∞ or $-\infty$.

Problem 3: (RLC Circuit)

Consider the series RLC -circuit shown in Figure 1, where x and y are the input and output, respectively.

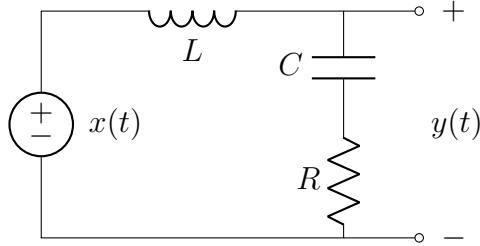


Figure 1: RLC circuit.

The transfer function $H(\omega)$, with the particular input and output signals specified in the figure above, is given by

$$H(\omega) = \frac{RC(i\omega) + 1}{LC(i\omega)^2 + RC(i\omega) + 1}. \quad (1)$$

Assume the system is initially at rest: $y(0^+) = \dot{y}(0^+) = 0$.

- (a) Determine the linear, constant-coefficient differential equation that governs the input-output behavior of the circuit.

Recall that an LCCDE has a frequency response of the following form:

$$H(\omega) = \frac{\sum_{m=0}^M b_m (i\omega)^m}{\sum_{n=0}^N a_n (i\omega)^n} \quad (2)$$

We match the given frequency response with the above equation.

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \dots a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \dots b_0 x(t) \quad (3)$$

If

$$b_0 = 1, b_1 = RC, a_0 = 1, a_1 = RC, a_2 = LC$$

then we arrive at the conclusion that

$$LC\ddot{y}(t) + RC\dot{y}(t) + y(t) = RC\dot{x}(t) + x(t). \quad (4)$$

(b) Using time domain analysis, derive the differential equation that governs the input-output behavior of the circuit. Does this match your answer from part a?

Hint: Here are some helpful equations for circuit analysis:

- For an inductor, $V_L = L \frac{dI_L}{dt}$.
- For a capacitor, $I_C = C \frac{dV_C}{dt}$.
- KCL: The sum of currents entering a node equals the sum of currents leaving the node.
- KVL: The sum of voltages in a closed loop is 0.

Applying KVL around the circuit:

$$x(t) - y(t) = V_L(t). \quad (5)$$

We know a capacitor is an energy storing unit.

$$i(t) = C \frac{dV_c}{dt}, \quad (6)$$

$$V_c(t) = y(t) - Ri(t). \quad (7)$$

Combining equations (6) and (7)

$$i(t) = C \frac{dy(t)}{dt} - RC \frac{di(t)}{dt}. \quad (8)$$

The inductor in the equation is governed by

$$\frac{V_L(t)}{L} = \frac{di(t)}{dt}. \quad (9)$$

Plugging equation (9) into equation (8):

$$\int \frac{V_L(t)}{L} dt = C \frac{dy(t)}{dt} - \frac{RC}{L} V_L(t), \quad (10)$$

$$\int V_L(t) dt = LC \frac{dy(t)}{dt} - RC V_L(t), \quad (11)$$

$$V_L(t) = \frac{L}{R} \frac{dy(t)}{dt} - \frac{1}{RC} \int V_L(t) dt. \quad (12)$$

Plugging equation (12) into equation (5):

$$x(t) - y(t) = \frac{L}{R} \frac{dy(t)}{dt} - \frac{1}{RC} \int V_L(t) dt, \quad (13)$$

$$\dot{x} - \dot{y} = \frac{L}{R} \ddot{y} - \frac{V_L}{RC}, \quad (14)$$

$$RC\dot{x} - RC\dot{y} = LC\ddot{y} - x(t) + y(t). \quad (15)$$

(c) Draw the integrator-adder-gain diagram for this system.

With all zero conditions, integrate both sides of (4) twice to convert the differential equation into an integrating form. Informally,

$$\begin{aligned} LCy + RC \int y + \iint y &= RC \int x + \iint x, \\ y &= \frac{R}{L} \int (x - y) + \frac{1}{LC} \iint (x - y). \end{aligned} \quad (16)$$

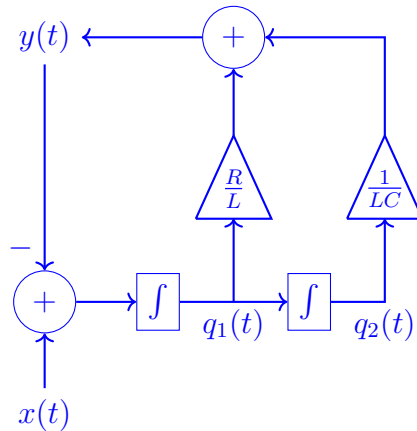


Figure 2: Integrator-adder-gain block diagram for an RLC circuit.

Note: states $q_1(t)$ and $q_2(t)$ are labeled for part (d) and onward.