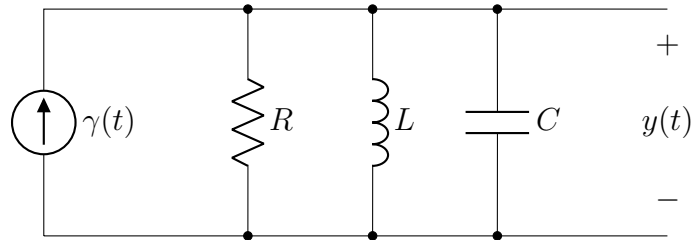


Problem 1: Parallel RLC Circuit

Consider the following parallel RLC circuit, driven by a current source $\gamma(t)$, with output $y(t)$ being the voltage across the parallel combination of R , L , and C . R , L , and C are all positive and real-valued.



By KCL at the top node, the source current splits between the capacitor ($i_C(t)$), the inductor ($i_L(t)$), and the resistor ($i_R(t)$):

$$\gamma(t) = i_R(t) + i_L(t) + i_C(t)$$

Note: You don't need to know how to read the circuit to answer this problem. The problem gives you all of the relevant equations.

The system dynamics are governed by Kirchhoff's Current Law (KCL) and the component constituent relations:

$$i_R(t) = \frac{y(t)}{R}, \quad i_C(t) = C\dot{y}(t), \quad y(t) = Li_L(t)$$

Define the following state variables:

1. $q_1(t) = i_L(t)$ (inductor current)
2. $q_2(t) = y(t)$ (output voltage)

Time-domain analysis of the circuit gives the following results:

$$\begin{aligned} \dot{i}_L(t) &= \frac{1}{L}y(t) \\ \dot{y}(t) &= -\frac{1}{C}i_L(t) - \frac{1}{RC}y(t) + \frac{1}{C}\gamma(t) \end{aligned}$$

(a) Find the state-space matrices **A**, **B**, **C**, and **D** for the continuous-time state-space representation:

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\gamma(t)$$

$$y(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\gamma(t)$$

From $y(t) = L \frac{di_L(t)}{dt}$, we express the derivative of $q_1(t)$ in terms of $q_2(t)$:

$$\dot{q}_1(t) = \frac{1}{L}q_2(t)$$

Substituting the constituent relations into KCL gives:

$$\gamma(t) = \frac{y(t)}{R} + i_L(t) + C \frac{dy(t)}{dt} = \frac{q_2(t)}{R} + q_1(t) + C\dot{q}_2(t)$$

Solving for $\dot{q}_2(t)$:

$$\dot{q}_2(t) = -\frac{1}{C}q_1(t) - \frac{1}{RC}q_2(t) + \frac{1}{C}\gamma(t)$$

Writing these differential equations in matrix form yields:

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} \gamma(t)$$

Since output $y(t) = q_2(t)$:

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + [0]\gamma(t)$$

Thus, the matrices are:

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{D} = [0]$$

(b) Consider the state matrix **A**. Then, answer the following questions:

- (i) Is the system internally/asymptotically stable? Justify your answer.

To find the eigenvalues of matrix $\mathbf{A}$, we solve the characteristic equation $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$:

$$\det \begin{bmatrix} \lambda & -\frac{1}{L} \\ \frac{1}{C} & \lambda + \frac{1}{RC} \end{bmatrix} = \lambda \left(\lambda + \frac{1}{RC} \right) + \frac{1}{LC} = \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{LC} = 0$$

Using the quadratic formula, the eigenvalues are:

$$\lambda_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

For physical component values $R, L, C > 0$, the real part of both eigenvalues $\text{Re}(\lambda_{1,2})$ is strictly negative:

$$\text{Re}(\lambda_{1,2}) < 0$$

Thus, the system is internally stable (and thus BIBO stable).

- (ii) Under what conditions (what values of R, L, C) is the system overdamped? Critically damped? Underdamped?

The eigenvalues are:

$$\lambda_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Damping cases: The behavior depends on the discriminant $\Delta = \left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}$:

- Overdamped: Occurs when Δ is positive: $R < \frac{1}{2}\sqrt{\frac{L}{C}}$. The eigenvalues $\lambda_{1,2}$ are real, distinct, and negative, resulting in pure exponential decay without oscillation.
- Critically damped: Occurs when Δ is zero: $R = \frac{1}{2}\sqrt{\frac{L}{C}}$. The eigenvalues are repeated, real, and negative ($\lambda_1 = \lambda_2 = -\frac{1}{2RC}$), yielding the fastest non-oscillatory settling to the equilibrium state.
- Underdamped: Occurs when Δ is negative: $R > \frac{1}{2}\sqrt{\frac{L}{C}}$. The eigenvalues form a complex conjugate pair with negative real parts, resulting in exponentially decaying sinusoidal oscillations.

- (iii) As R increases, how does the system's zero-input response change? Does the system become more heavily damped or less heavily damped?

As R increases, the real part of both eigenvalues approaches 0^- . As a result, the system becomes less heavily damped; the system's zero-input response will decay more slowly and will show oscillation for longer.

- (c) Find the frequency response $H(\omega)$ of the system using the eigenfunction property. Keeping in mind that $\dot{i}_L(t) = \frac{1}{L}y(t)$, start with the time-domain equation:

$$\gamma(t) = i_R(t) + i_L(t) + i_C(t) = \frac{1}{R}y(t) + i_L(t) + C\dot{y}(t)$$

Starting with the equation above, we differentiate to find $\dot{\gamma}(t)$:

$$\dot{\gamma}(t) = \frac{1}{R}\dot{y}(t) + \dot{i}_L(t) + C\ddot{y}(t)$$

Substituting $\dot{i}_L(t) = \frac{y(t)}{L}$ yields the governing differential equation:

$$C\ddot{y}(t) + \frac{1}{R}\dot{y}(t) + \frac{1}{L}y(t) = \dot{\gamma}(t)$$

By the eigenfunction property of LTI systems, for input $\gamma(t) = e^{j\omega t}$, the output takes the form $y(t) = H(\omega)e^{j\omega t}$. Computing the necessary derivatives:

$$\dot{\gamma}(t) = j\omega e^{j\omega t}$$

$$\dot{y}(t) = j\omega H(\omega)e^{j\omega t}$$

$$\ddot{y}(t) = (j\omega)^2 H(\omega)e^{j\omega t}$$

Substituting these into the differential equation:

$$C(j\omega)^2 H(\omega)e^{j\omega t} + \frac{1}{R}(j\omega)H(\omega)e^{j\omega t} + \frac{1}{L}H(\omega)e^{j\omega t} = j\omega e^{j\omega t}$$

Factoring out $e^{j\omega t}$ and solving for $H(\omega)$:

$$\left(C(j\omega)^2 + \frac{1}{R}j\omega + \frac{1}{L} \right) H(\omega) = j\omega$$

$$H(\omega) = \frac{j\omega}{C(j\omega)^2 + \frac{j\omega}{R} + \frac{1}{L}} = \frac{\frac{j\omega}{C}}{\left(\frac{1}{LC} - \omega^2\right) + \frac{1}{R} \frac{j\omega}{C}} = \frac{\omega}{\frac{C}{j} \left(\frac{1}{LC} - \omega^2\right) + \frac{1}{R}\omega}$$

$$= \boxed{R \cdot \frac{\omega}{\omega + jRC \left(\omega^2 - \frac{1}{LC}\right)}}$$

(d) Find the magnitude $|H(\omega)|$ and phase $\angle H(\omega)$ of the frequency response. Sketch both plots, indicating key values on the axes, and state what type of filter this circuit represents.

From the previous part:

$$H(\omega) = R \cdot \frac{\omega}{\omega + jRC \left(\omega^2 - \frac{1}{LC}\right)}$$

Thus, the magnitude is:

$$|H(\omega)| = R \cdot \frac{\omega}{\sqrt{\omega^2 + (RC)^2 \left(\omega^2 - \frac{1}{LC}\right)^2}}$$

The magnitude reaches its maximum value of R at the resonant frequency $\omega = \pm\omega_0 = \pm\frac{1}{\sqrt{LC}}$, and approaches 0 as $\omega \rightarrow 0$ and $|\omega| \rightarrow \infty$.

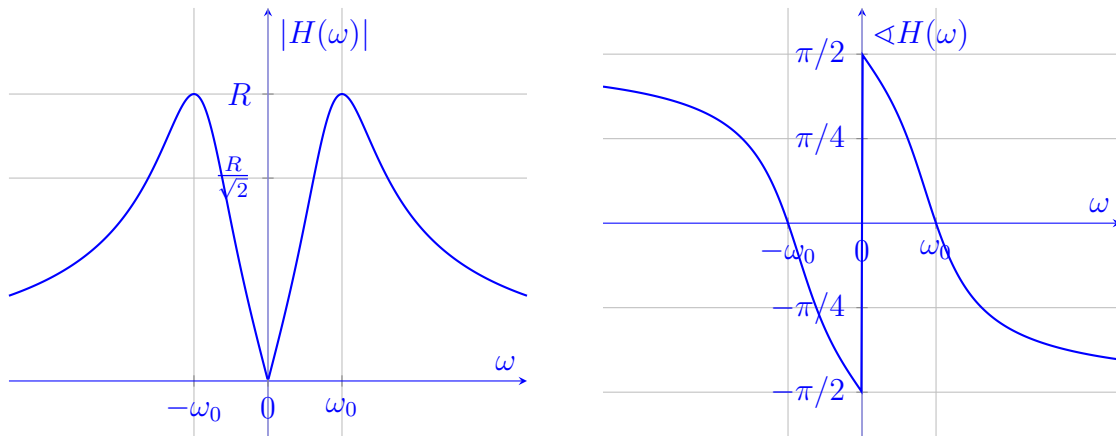
An alternative form of the frequency response:

$$H(\omega) = \frac{R}{1 + jR \left(\omega C - \frac{1}{\omega L}\right)}$$

The phase is:

$$\angle H(\omega) = -\arctan \left(R \left(\omega C - \frac{1}{\omega L} \right) \right)$$

The phase is an odd function of ω . As $\omega \rightarrow 0^+$, $\angle H(\omega) \rightarrow \frac{\pi}{2}$; at resonance $\omega = \omega_0$, $\angle H(\omega_0) = 0$; and as $\omega \rightarrow \infty$, $\angle H(\omega) \rightarrow -\frac{\pi}{2}$.



The circuit attenuates frequencies near DC ($\omega \rightarrow 0$) and high frequencies ($|\omega| \rightarrow \infty$) while passing frequencies around the resonant frequency $\omega_0 = \frac{1}{\sqrt{LC}}$, so it is a band-pass filter.

- (e) Draw an Integrator-Adder-Gain (IAG) block diagram representing this system. Clearly label the state variables $q_1(t)$ and $q_2(t)$.

Recall the state equations and output equation from Part (a):

$$\begin{aligned}\dot{q}_1(t) &= \frac{1}{L}q_2(t) \\ \dot{q}_2(t) &= \frac{1}{C}\gamma(t) - \frac{1}{C}q_1(t) - \frac{1}{RC}q_2(t) \\ y(t) &= q_2(t)\end{aligned}$$

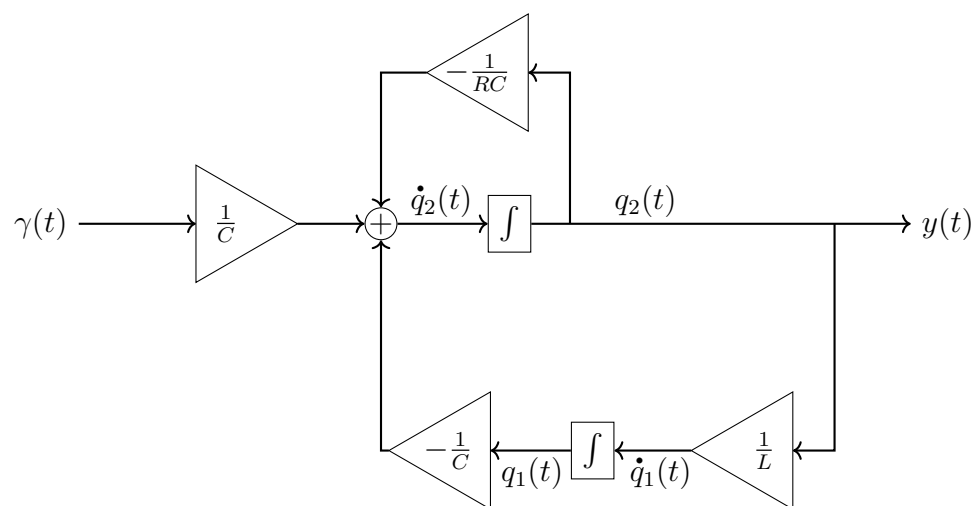


Figure 1: Integrator-adder-gain block diagram for the parallel RLC state-space system