

Problem 1: Feedback Stabilization

Consider the well-structured interconnection H of causal discrete-time LTI systems A and B , as shown in Figure 1. Each individual system is defined on $[\mathbb{Z} \rightarrow \mathbb{C}] \rightarrow [\mathbb{Z} \rightarrow \mathbb{C}]$.

- (a) Determine the composite system's frequency response $H : \mathbb{R} \rightarrow \mathbb{C}$ in terms of the frequency responses A and B of the individual components. Assume the existence of the frequency response.

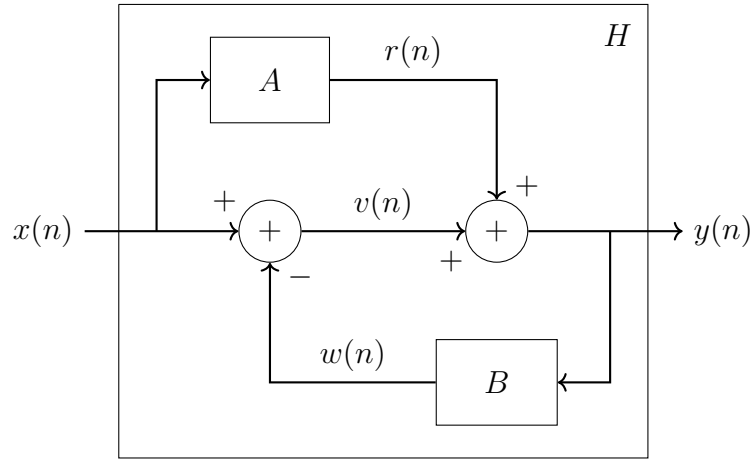


Figure 1: Interconnected DT-LTI system.

As usual, to find the frequency response, we let our input $x(n) = e^{i\omega n}$ and the output be $H(\omega)e^{i\omega n}$. From here, we can find the intermediary signals $r(n)$ and $w(n)$.

$$\begin{aligned} r(n) &= A(\omega)e^{i\omega n} \\ w(n) &= B(\omega)H(\omega)e^{i\omega n} \end{aligned}$$

Once we have $w(n)$, we can find the intermediary signal $v(n)$.

$$v(n) = x(n) - w(n) = e^{i\omega n} - B(\omega)H(\omega)e^{i\omega n}$$

Finally, we find the output $y(t)$.

$$y(n) = r(n) + v(n)$$

$$H(\omega)e^{i\omega n} = A(\omega)e^{i\omega n} + e^{i\omega n} - B(\omega)H(\omega)e^{i\omega n}$$

$$(1 + B(\omega))H(\omega) = 1 + A(\omega)$$

$$H(\omega) = \frac{1 + A(\omega)}{1 + B(\omega)}$$

For the remaining parts, suppose the systems A and B have impulse response values

$$a[n] = \delta[n - 1] \qquad b[n] = -3\delta[n - 1]$$

(b) Is the system H stable?

$$\begin{aligned} w[n] &= (b * h)[n] \\ v[n] &= \delta[n] - (b * h)[n] \\ h[n] &= a[n] + \delta[n] - (b * h)[n] \\ \Rightarrow (\delta * h)[n] + (b * h)[n] &= a[n] + \delta[n] \\ ((\delta + b) * h)[n] &= a[n] + \delta[n] \end{aligned}$$

$$\text{LHS} = ((\delta + b) * h)[n] = h[n] - 3h[n - 1]$$

$$\text{RHS} = a[n] + \delta[n] = \delta[n] + \delta[n - 1]$$

LCCDE:

$$h[n] - 3h[n - 1] = \delta[n] + \delta[n - 1]$$

Use superposition:

$$h_0[n] - 3h_0[n - 1] = \delta[n]$$

$$h_1[n] - 3h_1[n - 1] = \delta[n - 1]$$

$$h_1[n] = h_0[n - 1] \text{ by TI} \Rightarrow h[n] = h_0[n] + h_1[n] = 3^n u[n] + 3^{n-1} u[n - 1]$$

(c) Suppose we put H in negative feedback with a constant gain K as shown in fig. 2. For what values of K will the resulting system G be stable? Under this assumption G has a frequency response, so we can use frequency-response arguments (Black's formula).

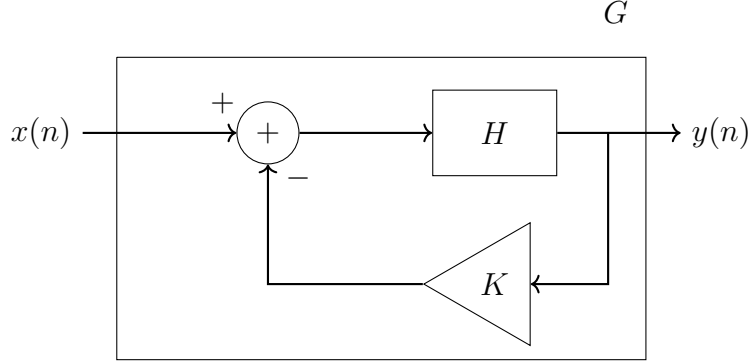


Figure 2: H in negative feedback with a constant gain

By Black's formula,

$$G(\omega) = \frac{H(\omega)}{1 + KH(\omega)} = \frac{1 + e^{-i\omega}}{1 - 3e^{-i\omega} + K + Ke^{-i\omega}} = \frac{1 + e^{-i\omega}}{(1 + K) - (3 - K)e^{-i\omega}}$$

Writing this as an LCCDE, we get

$$(1 + K)y(n) - (3 - K)y(n - 1) = x(n) + x(n - 1).$$

We can find the impulse response by letting $y(n) = h(n)$ and $x(n) = \delta(n)$.

$$h(n) = \frac{1}{1 + K}\delta(n) + \frac{1}{1 + K}\delta(n - 1) + \frac{3 - K}{1 + K}h(n - 1).$$

$h(n) = 0$ when $n < 0$ due to causality.

$$h(0) = \frac{1}{K + 1}$$

$$h(1) = \frac{1}{K + 1} + \frac{3 - K}{(K + 1)^2} = \frac{4}{(K + 1)^2}$$

$$h(2) = \frac{3 - K}{K + 1} \cdot \frac{4}{(K + 1)^2}$$

$$h(3) = \left(\frac{3 - K}{K + 1}\right)^2 \cdot \frac{4}{(K + 1)^2}$$

$\vdots$

$$\therefore h(n) = \frac{1}{K + 1}\delta(n) + \frac{4}{(K + 1)^2} \left(\frac{3 - K}{K + 1}\right)^{n-1} u(n - 1)$$

This impulse response is absolutely summable when the geometric series $\left(\frac{3-K}{K+1}\right)$ converges, which requires

$$\left|\frac{3-K}{K+1}\right| < 1$$

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$$\implies |3-K| < |K+1|$$

When $K = 1$, $|3-K| = |K+1|$, and for $K > 1$, the inequality holds, while for $K < 1$, the inequality does not hold. Hence, the system is stable when $K > 1$.

Notice what we have done here: we made an unstable system stable just by placing it in negative feedback with an appropriate constant gain! The idea that this is possible forms the basis of Feedback Control, the topic of EE 128.

Problem 2: DTFS and Change of Basis

In this problem, we will build intuition on Fourier transforms by looking at the DTFS through the lens of change of basis.

We chose the DTFS for this problem because discrete, periodic signals lend themselves well to the vector operations many of you have seen in basic linear algebra. If we look at the space of discrete-time signals with period N , we can represent each signal as a vector $\in \mathbb{R}^N$ consisting of one period of the signal.

- (a) To familiarize yourself with this vector notation, find the fundamental period N of $x[n] = \cos[\pi n] + \sin[\frac{\pi}{2}n]$ and write $x[n]$ as a vector, $\mathbf{x}_v$, in $\mathbb{R}^N$.

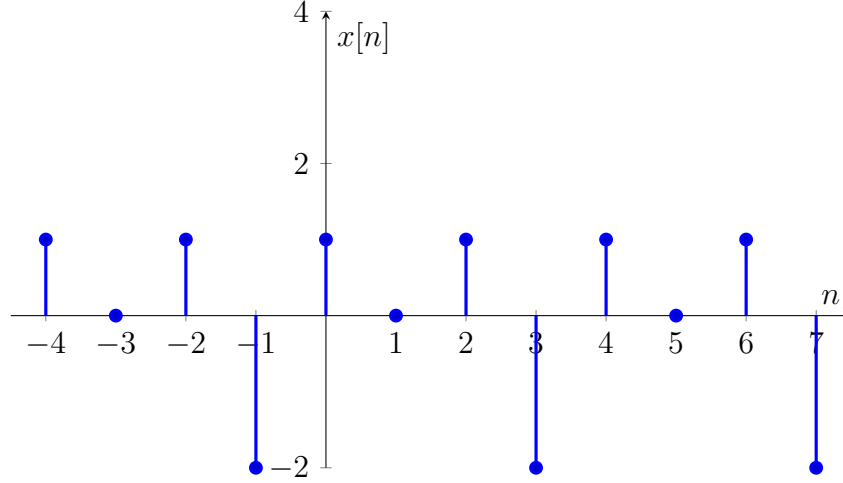
The period of $x[n]$ is the smallest N such that

$$x[n + N] = \cos[\pi(n + N)] + \sin[\frac{\pi}{2}(n + N)] = x[n].$$

More specifically, we need to choose the smallest integer N such that πN and $\frac{\pi}{2}N$ are both integer multiples of 2π . So, the fundamental period of $x[n]$ is $N = 4$.

If we evaluate $x[n]$ at every timestep within one period, we get the following vector representation for x :

$$\mathbf{x}_v = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}.$$



(b) We would like to define an orthogonal basis of complex exponentials for $\mathbb{R}^N$ by using vectors of the form $\mathbf{v}_k[n] = e^{i\omega_k n}$. Remember, we are working in the space of N -periodic functions, so $\mathbf{v}[n]$ must correspond to a complex exponential with period N .

For which values of ω_k does $e^{i\omega_k n}$ have a period of N ? For an arbitrary period N , how many unique vectors of this form exist?

For a complex exponential of the form $e^{i\omega_k n}$ to have a period of N , $\omega_k N$ has to be an integer multiple of 2π . In other words, $\omega_k = \frac{2\pi}{N}k$ for integer k .

Unlike in continuous time, frequencies in discrete time are 2π -periodic. A result of this is that there are only N unique frequencies of the form $\frac{2\pi}{N}k$. To show why, consider a complex exponential with frequency $\frac{2\pi}{N}$:

$$e^{i\frac{2\pi}{N}(k+N)n} = e^{i\frac{2\pi}{N}kn} e^{2\pi n} = e^{i\frac{2\pi}{N}kn}.$$

This implies that the frequency $\frac{2\pi}{N}(k+N)$ is equivalent to the frequency $\frac{2\pi}{N}k$, so there must only be N unique frequencies of the form $\frac{2\pi}{N}k$.

Intuitively, for example $N = 4$, signals with period of 1, 2, 3, 4, add up together; the summed signal has period of 4, so for 4-periodic signal space, we need 4 unique basis vectors. $e^{i\omega_k n}$ has a period of N for $\omega_k = \frac{2\pi}{N}k$, $k \in \langle N \rangle$.

(c) Show that the set vectors $\{\mathbf{v}_k = e^{i\omega_k n}\}$ using the values of ω_k found in the previous part are orthogonal to each other. A set of vectors is defined as being orthog-

onal if $\mathbf{v}_l^* \mathbf{v}_m = 0$, $\forall l \neq m$, and the inner product of two complex-valued vectors is defined as follows:

$$\mathbf{a}^* \mathbf{b} = \sum_{n=0}^{N-1} \bar{a}[n] b[n],$$

where $\mathbf{a}^*$ is the conjugate transpose of $\mathbf{a}$ and $\bar{a}[n]$ is the complex conjugate of $a[n]$.

Hint: the formula for the summation of a geometric series is

$$\sum_{k=0}^{N-1} a^k = \frac{a^N - 1}{a - 1}.$$

Recall that, in the previous part, we found

$$\omega_k = \frac{2\pi}{N}k, \quad k \in \langle N \rangle.$$

Using the definition of orthogonality in the problem statement, let us evaluate the inner product of two arbitrary vectors in the set $\{\mathbf{v}_k\}$:

$$\begin{aligned} \mathbf{v}_l^* \mathbf{v}_m &= \sum_{n=0}^{N-1} e^{-i\omega_l n} e^{i\omega_m n} \\ &= \sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}(l-m)n}. \end{aligned}$$

We can apply the formula for a geometric summation to get

$$\mathbf{v}_l^* \mathbf{v}_m = \frac{e^{-i\frac{2\pi}{N}(l-m)N} - 1}{e^{-i\frac{2\pi}{N}(l-m)} - 1}.$$

If $l \neq m$,

$$\mathbf{v}_l^* \mathbf{v}_m = \frac{e^{-i2\pi(l-m)} - 1}{e^{-i\frac{2\pi}{N}(l-m)} - 1} = \frac{1 - 1}{e^{-i\frac{2\pi}{N}(l-m)} - 1} = 0.$$

So, the vectors in $\{\mathbf{v}_k\}$ are orthogonal.

(d) What is the norm of $\mathbf{v}_k$? The norm of a vector is the square root of the inner product of that vector with itself.

First find the norm squared of $\mathbf{v}_k$, and then take the square root to get $\|\mathbf{v}_k\|$.

$$\|\mathbf{v}_k\|^2 = \mathbf{v}_k^* \mathbf{v}_k = \sum_{n=0}^{N-1} e^{-i\omega_k n} e^{i\omega_k n} = \sum_{n=0}^{N-1} 1 = N.$$

So, the norm of $\mathbf{v}_k$ is $\sqrt{N}$.