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Problem 1: Mystery DTFS

Consider a periodic real-valued discrete-time signal x having fundamental period p and an exponential discrete Fourier series expansion

$$x[n] = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} \tag{1}$$

where ω_0 is the fundamental frequency of the signal. The signal has the following properties:

- $x[n+3] = x[n], \forall n \in \mathbb{Z}$
- $\sum_{n \in \langle p \rangle} x[n] = 0$
- $\sum_{k \in \langle p \rangle} X_k = 0$
- $\sum_{k \in \langle p \rangle} |X_k|^2 = \frac{1}{2}$

Show that the signal x can be expressed as $x[n] = A\cos(Bn + C)$, and determine the parameters A, B, and C.

- The first hint tells us the signal has a period $p = 3$. Without loss of generality, let $\langle p \rangle = \{-1, 0, 1\}$.
- The quantity in the third hint is actually $x[0]$ in disguise.
- The quantity in the second hint is actually pX_0 in disguise—refer to the DTFS analysis equation. The implication is that $X_0 = 0$ and $X_{-1} = -X_1$.

At this point, we should pattern match the DTFS expansion of x with the form x should be in.

$$\begin{aligned} x[n] &= X_{-1}e^{-i\omega_0 n} + X_1 e^{i\omega_0 n} && (2)\\ &= X_1\left(e^{i\omega_0 n} - e^{-i\omega_0 n}\right) && (3)\\ &= 2iX_1 \sin(\omega_0 n) && (4)\\ &= A\cos(Bn + C) && (5) \end{aligned}$$

Setting $B = \omega_0 = 2\pi/3$, $C = -\pi/2$, and $A = 2iX_1$ should bring the expansion into the form we would like. Finally, using the fourth hint,

$$|X_{-1}|^2 + |X_1|^2 = 2|X_1|^2 = \frac{1}{2} \implies |X_1| = \frac{1}{2} \tag{6}$$

Due to conjugate symmetry, $X_{-1} = X_1^\star$, so $\mathrm{Re}(X_{-1}) = \mathrm{Re}(X_1)$. However, since $X_{-1} = -X_1$, we also know $\mathrm{Re}(X_{-1}) = -\mathrm{Re}(X_1)$. Therefore, $\mathrm{Re}(X_1) = 0$, X_1 and X_{-1} are purely imaginary.

Note that the choice for X_1 is not unique, since $X_1 = -i/2$ and $X_1 = i/2$ satisfy $|X_1| = 1/2$ and $\mathrm{Re}(X_1) = 0$. For $X_1 = -i/2$, $A = 1$, while for $X_1 = i/2$, $A = -1$.

Problem 2: Proving DTFS Properties

Given a periodic signal x with period p and frequency $\omega_0 = 2\pi/p$, recall the synthesis and analysis equations of DTFS:

$$x[n] = \sum_{k\in\langle p\rangle} X_k e^{ik\omega_0 n}$$

$$X_k = \frac{1}{p}\sum_{n\in\langle p\rangle} x[n]e^{-ik\omega_0 n}$$

Using these equations, prove the following properties:

(a) Conjugate Symmetry: The DTFS coefficients are conjugate symmetric —that is, $X_k = X^*_{-k}$ —if, and only if, x is a real-valued signal.

If x is a real valued signal, then $X_k = X^*_{-k}$:

Using the analysis equation,

$$\begin{aligned} X^*_{-k} &= \left(\frac{1}{p}\sum_{n\in\langle p\rangle} x[n]e^{-i(-k)\omega_0 n}\right)^* \\ &= \frac{1}{p}\sum_{n\in\langle p\rangle} x^*[n]\left(e^{ik\omega_0 n}\right)^* \\ &= \frac{1}{p}\sum_{n\in\langle p\rangle} x[n]e^{-ik\omega_0 n} \\ &= X_k \end{aligned}$$

If $X_k = X^*_{-k}$, then x is a real valued signal:

Using the analysis equation,

$$\begin{aligned} X_k &= X^*_{-k} \\ \frac{1}{p}\sum_{n\in\langle p\rangle} x[n]e^{-ik\omega_0 n} &= \left(\frac{1}{p}\sum_{n\in\langle p\rangle} x[n]e^{-i(-k)\omega_0 n}\right)^* \\ \sum_{n\in\langle p\rangle} x[n]e^{-ik\omega_0 n} &= \sum_{n\in\langle p\rangle} x^*[n]e^{-ik\omega_0 n} \end{aligned}$$

Since $e^{-ik\omega_0 n}$ are linearly independent for $n \in \langle p \rangle$, corresponding terms must be equal and $x[n] = x^*[n]$.

(b) Evenness: The DTFS coefficients are real and even —that is, $X_k = X_{-k} = X_k^*$ —if, and only if, x is a real-valued and even signal.

If $x[n]$ is real valued and even ($x[n] = x[-n]$), then $X_k = X_{-k} = X_k^*$:

Applying conjugate symmetry from part (a): $X_{-k} = X_k^*$.

Using the synthesis equation:

$$\begin{aligned} x[n] &= x[-n] \\ \sum_{k\in\langle p\rangle} X_k e^{ik\omega_0 n} &= \sum_{k\in\langle p\rangle} X_k e^{ik\omega_0[-n]} \\ &= \sum_{k\in\langle p\rangle} X_k e^{i(-k)\omega_0 n} \\ &= \sum_{k\in\langle p\rangle} X_{-k} e^{ik\omega_0 n} \end{aligned}$$

Since $e^{ik\omega_0 n}$ are linearly independent for $k \in \langle p \rangle$, corresponding terms must be equal and $X_k = X_{-k}$

If $X_k = X_{-k} = X_k^*$, then $x[n]$ is real valued and even ($x[n] = x[-n]$):

Applying conjugate symmetry from part (a): $x[n]$ must be real-valued.

Using the analysis equation:

$$
\begin{aligned}
X_k &= X_{-k} \\
\frac{1}{p}\sum_{n\in\langle p\rangle} x[n]e^{-ik\omega_0 n} &= \frac{1}{p}\sum_{n\in\langle p\rangle} x[n]e^{-i(-k)\omega_0 n} \\
&= \frac{1}{p}\sum_{n\in\langle p\rangle} x[-(-n)]e^{i(-k)\omega_0[-n]} \\
&= \frac{1}{p}\sum_{n\in\langle p\rangle} x[-n]e^{i(-k)\omega_0 n}
\end{aligned}
$$

Since $e^{-ik\omega_0 n}$ are linearly independent for $n \in \langle p\rangle$, corresponding terms must be equal and $x[n] = x[-n]$.

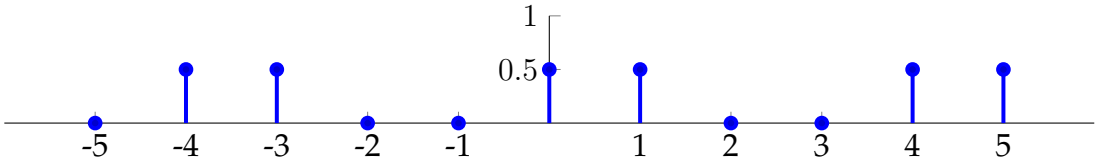
Problem 3: What did DTFS know about H(ω)?

Consider the two-point moving average filter **H** whose impulse response h and frequency response H are given by:

$$h[n] = \frac{\delta[n] + \delta[n-1]}{2} \text{ and } H(\omega) = \cos(\omega/2)e^{-i\omega/2}$$

Now suppose we construct a discrete-time function g by a periodic replication of h, according to the following: $g[n] = \sum_{l=-\infty}^{\infty} h[n - lN]$, for all integers n and for some integer $N \geq 2$.

(a) Plot $g[n]$ for $N = 4$.



(b) Determine a reasonably simple expression for the coefficient G_k in the discrete-time Fourier series expansion of g. Your expression for G_k must show how the DTFS coefficients relate to the frequency response $H(\omega)$. What do you notice as $N \to \infty$?

We see that:

$$\begin{aligned} G_k &= \frac{1}{N}\sum_{n=0}^{N-1} g[n]e^{-ik\frac{2\pi}{N}n} \\ &= \frac{1}{N}\sum_{n=0}^{1} g[n]e^{-ik\frac{2\pi}{N}n} \\ &= \frac{1}{N}\sum_{n=0}^{1} h[n]e^{-ik\frac{2\pi}{N}n} \\ &= \frac{1}{N}H(\omega)|_{\omega=\frac{k2\pi}{N}=k\omega_0} \end{aligned}$$

Where $\omega_0 = \frac{2\pi}{N}$, the fundamental frequency. We see that the DTFS coefficients *sample* $H(\omega)$ at multiples of the the fundamental frequency. As $N \to \infty$, the sampling

occurs at higher and higher resolution, and the coefficients G_k slowly approach $H(\omega)$. This phenomenon will become clear once you learn about the Discrete Time Fourier Transform (DTFT).

DTFS Properties

The Discrete Time Fourier Series is a method of analyzing periodic discrete time signals. In particular, the idea is that discrete time periodic signals can be expressed as a linear combination of the roots of unity corresponding to the fundamental period of the signal.

Suppose we have a signal $x[n]$ such that $x[n+p] = x[n]\,\forall n$. Then, x is said to be p periodic. The smallest such p is the fundamental period (denoted as p_x, or N), and the fundamental frequency is $\omega_0 = \frac{2\pi}{p_x}$. We choose one period of samples using the notation $\langle p \rangle$ to denote an interval of p integers (for example, $[0, 1, 2, \ldots, p-1]$).

Key Concept: Synthesis Equation: The synthesis equation says that we can decompose x into a linear combination of the Nth roots of unity. Specifically,

$$x[n] = \sum_{k \in \langle p_x \rangle} X_k e^{ik\omega_0 n}$$

The DTFS coefficients X_k can be calculated through the **Analysis Equation**:

$$X_k = \frac{1}{p_x} \sum_{n \in \langle p_x \rangle} x[n] e^{-ik\omega_0 n}$$

A natural consequence of the analysis equation is that the DTFS coefficients are also p-periodic; that is, $X_k = X_{k+p}$.

Properties of the DTFS:

1. Conjugate Symmetry: $x[n]$ is a real valued signal if and only if $X_k = X_{-k}^*$
2. Evenness: $x[n]$ is real valued and even ($x[n] = x[-n]$) if and only if $X_k = X_{-k} = X_k^*$, aka the DTFS coefficients are also real valued and even.
3. Oddness: $x[n]$ is real valued and odd ($x[n] = -x[-n]$) if and only if $X_k = -X_{-k} = -X_k^*$, aka the DTFS coefficients are also imaginary valued and odd.
4. Poisson's identity: $x[n] = \sum_{l=-\infty}^{\infty} \delta[n - lp] = \frac{1}{p} \sum_{k=0}^{p-1} e^{ik\omega_0 n}$ where $\omega_0 = \frac{2\pi}{p}$. This says that if x is an impulse train in time, then $X_k = \frac{1}{p}$.
5. Linearity: $\alpha x[n] \iff \alpha X_k$, and $x[n] + y[n] \iff X_k + Y_k$
6. Parseval's Theorem: $\frac{1}{N} \sum_{\langle N \rangle} |x[n]|^2 = \sum_{k=0}^{N-1} |X_k|^2$