UNIVERSITY OF CALIFORNIA BERKELEY
Department of EECS

**Problem 1: CTFS Practice**

(a) Find the Fourier series coefficients $X_k$ for the following signal. Use the minimum period.

$$x(t) = \sin(\pi t) + 2\cos(3\pi t)$$

The periods of the two constituent sinusoids are $2$ and $\frac{2}{3}$. Since the latter fits into the former an integer number of times, we may assume that $x(t)$ has a period that divides LCM $\left(2, \frac{2}{3}\right) = 2$. We can work with a period of $2$, and the coefficients will still be accurate (an overestimate by an integer factor is fine, since the extraneous coefficients will come out to $0$ in the analysis equation, and the true frequencies will still be present). The fundamental frequency is $\omega_0 = \frac{2\pi}{T} = \pi$. We can decompose $x(t)$ into complex exponentials:

$$x(t) = \frac{1}{2i}(e^{i\pi t} - e^{-i\pi t}) + 2\left(\frac{1}{2}(e^{i3\pi t} + e^{-i3\pi t})\right)$$

$$X_k = \begin{cases} -\frac{1}{2i} & k = -1 \\ \frac{1}{2i} & k = +1 \\ 1 & k = \pm 3 \end{cases}$$

(b) Let $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$ be a $p$-periodic signal.

(i) Determine the Fourier coefficients of $y(t) = x(t - T)$, for some $T \in \mathbb{R}$.

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0(t-T)} \\ &= \sum_{k=-\infty}^{\infty} \left(X_k e^{-ik\omega_0 T}\right) e^{ik\omega_0 t} \\ &\implies Y_k = X_k e^{-ik\omega_0 T} \end{aligned}$$

(ii) Let $z(t) = e^{iM\omega_0 t}x(t)$, for some $M \in \mathbb{Z}$. Determine the Fourier coefficients of $z(t)$.

$$z(t) = e^{iM\omega_0 t}\sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$
$$= \sum_{k=-\infty}^{\infty} X_k e^{i(k+M)\omega_0 t}$$

By a change of variables, $l = k + M$, we have

$$= \sum_{l=-\infty}^{\infty} X_{l-M} e^{il\omega_0 t}$$
$$\implies Z_l = X_{l-M}$$

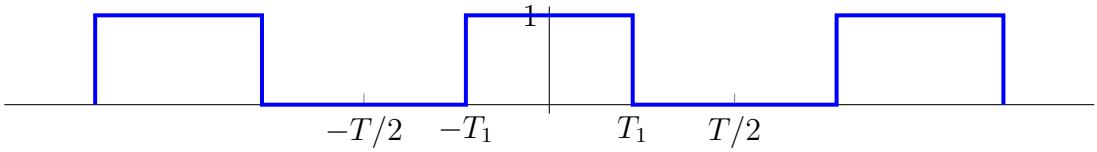
(c) Given the Fourier series coefficients, determine the continuous time signal $x(t)$ with period $T = 4$:

$$X_k = \begin{cases} ik & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \sum_{k=-2}^{2} ike^{ik\frac{\pi}{2}t} = ie^{i\frac{\pi}{2}t} - ie^{-i\frac{\pi}{2}t} + 2ie^{i\pi t} - 2ie^{-i\pi t}$$
$$= i\left(2i\sin\left(\frac{\pi}{2}t\right)\right) + 2i\left(2i\sin\left(\pi t\right)\right)$$
$$= -2\sin\left(\frac{\pi}{2}t\right) - 4\sin\left(\pi t\right)$$

(d) Find the Fourier coefficients $X_k$ for the following signal. Assume $x(t)$ is periodic with period $T$.

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & T_1 < |t| \le \frac{T}{2} \end{cases}$$




We divide our coefficients into two cases. Where $k \neq 0$:

$$\begin{aligned} X_k &= \frac{1}{T}\int_{<T>} x(t)e^{-ik\omega_0 t}dt \\ &= \frac{1}{T}\int_{-T_1}^{T_1} e^{-ik\omega_0 t}dt \\ &= \frac{1}{T}\frac{1}{-ik\omega_0}e^{-ik\omega_0 t}\Big|_{-T_1}^{T_1} \\ &= \frac{2}{Tk\omega_0}\frac{1}{2i}[-e^{-ik\omega_0 t} + e^{ik\omega_0 t}] \\ &= \frac{1}{k\pi}\sin(k\omega_0 T_1) \end{aligned}$$

And for when $k = 0$:

$$X_0 = \frac{1}{T}\int_{-T_1}^{T_1} e^0 dt = \frac{1}{T}t\Big|_{-T_1}^{T_1} = \frac{2T_1}{T}$$

**Problem 2: DTFT Multiplication Property**

A discrete-time signal and its Fourier transform are described below:

$$g(n) = \frac{B}{\pi n}\sin(An) \overset{\mathcal{F}}{\longleftrightarrow} G(\omega) = \begin{cases} B & \text{if } |\omega| < A \\ 0 & \text{if } |\omega| \geq A \end{cases}$$

where $A, B \in \mathbb{R}$. It is understood $G(\omega)$ is $2\pi$-periodic, and that $0 < A < \pi/2$.

Consider a related signal $h(n) = (g(n))^2$ for all $n \in \mathbb{Z}$.

(a) Plot and label $H(\omega)$, the DTFT of $h$.

*Hint:* recall that the multiplication property says

$$x(n)\, y(n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}(X \circledast Y)(\omega) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} X(\lambda)\, Y(\omega - \lambda)\, d\lambda$$

for any $x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ and $y(n) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega)$.

Applying the multiplication property,

$$g(n)\, g(n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{\langle 2\pi \rangle} G(\lambda)\, G(\omega - \lambda)\, d\lambda$$

For circular convolution, we keep one (periodic) function intact, keep only one period of the other function, and then perform regular convolution. Let $\widetilde{G}(\omega)$ denote one period of $G(\omega)$.

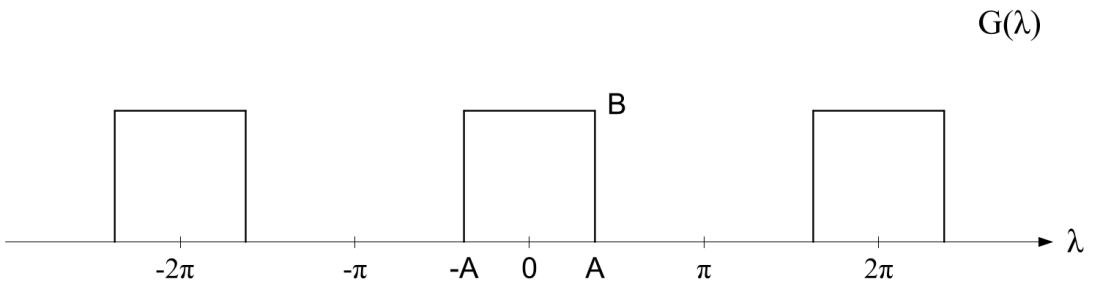



Figure 1: $G(\lambda)$

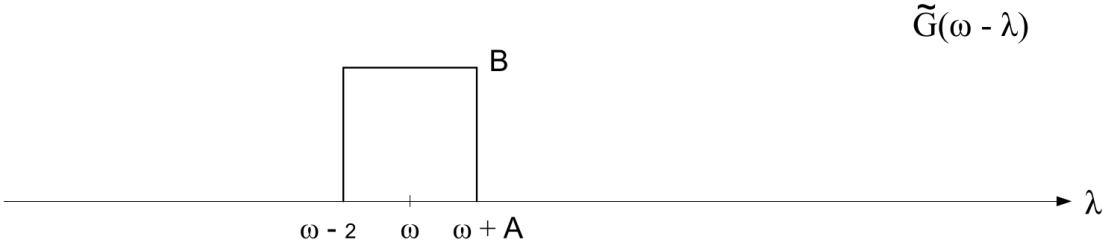



Figure 2: $\widetilde{G}(\omega - \lambda)$

Convolving two rectangles yields a triangle whose maximum occurs when the boxes completely overlap (maximum overlapping area).

$$\frac{1}{2\pi}\int_{-A}^{A} B^2 d\lambda = \frac{2AB^2}{2\pi} = \frac{AB^2}{\pi}$$

(b) If $H$ is the frequency response of a filter, and the input to the filter is

$$x(n) = 1 + \cos(An)$$

determine the corresponding output $y$.

Recall that complex exponentials $\{e^{j\omega n} : \omega \in \mathbb{R}\}$ are eigenfunctions of LTI systems. We can rewrite $x(n)$ as

$$x(n) = e^{j(0)\,n} + \frac{1}{2}e^{j(A)\,n} + \frac{1}{2}e^{j(-A)n}$$

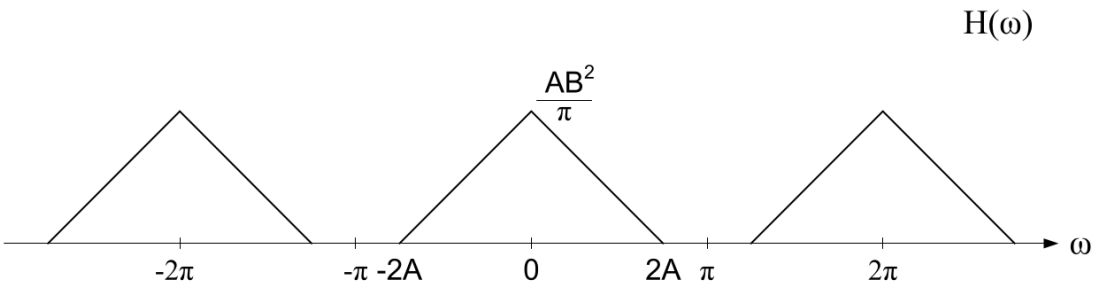



Figure 3: $H(\omega)$

Using linearity, $H(\omega)$ scales each exponential accordingly.

$$\begin{aligned} y(n) &= H(0)\, e^{j(0)\, n} + \frac{1}{2} H(A)\, e^{j(A)\, n} + \frac{1}{2} H(-A)\, e^{j(-A) n} \\ &= \frac{AB^2}{\pi} + \frac{1}{2}\frac{AB^2}{2\pi} e^{j(A)\, n} + \frac{1}{2}\frac{AB^2}{2\pi} e^{j(-A)\, n} \\ &= \frac{AB^2}{\pi}\left(1 + \frac{1}{2}\cos(An)\right) \end{aligned}$$

**Problem 3: Scaling, Revisited**

Define the *area* of a discrete-time signal $x$ as

$$A_x = \sum_{n=-\infty}^{\infty} x(n)$$

for all signals such that the sum converges. You may assume all signals in this question have finite area.

(a) Let $y(n) = (x * h)(n)$. Using the DTFT analysis equation, show that $A_y = A_x A_h$.

First, note that

$$X(0) = \sum_{n=-\infty}^{\infty} x(n) = A_x$$

Next, since convolution in time is multiplication in frequency, we have

$$Y(\omega) = X(\omega)H(\omega) \implies Y(0) = X(0)H(0) \implies A_y = A_x A_h$$

as desired.

(b) Given a DT-LTI filter $H$ with impulse response $h$, describe how to construct an equivalent[1] filter $G$ with impulse response $g$ that preserves the area of its inputs. That is, if $y$ is the output of $G$ to $x$, then it should hold that $A_y = A_x$.

To construct $g$ from $h$, simply divide $h$ by the sum of its coefficients. More formally, we can define

$$g(n) = \frac{1}{A_h} h(n)$$

Using the result of part (a), if we send $x$ into $G$ and receive $y$, then

[1] For this problem, we say that two filters are equivalent if we can perform a simple operation, such as scaling by a constant or adding or subtracting a constant, to convert between their impulse responses.

$$
\begin{aligned}
A_y &= A_x A_g \\
&= A_x \sum_{n=-\infty}^{\infty} g(n) \\
&= A_x \sum_{n=-\infty}^{\infty} \frac{1}{A_h} h(n) \\
&= \frac{A_x}{A_h} \sum_{n=-\infty}^{\infty} h(n) \\
&= \frac{A_x}{A_h} A_h \\
&= A_x
\end{aligned}
$$

This technique is easily applicable in practice and is often used in scientific simulations when the signals measured represent power (in which case their areas represent energy, so applying this normalization corresponds to preserving the energy of the input signal).

In Python, for example, if we had access to the coefficients of $h$ as a numpy array, we could obtain the coefficients of $g$ with the code `g = h / np.sum(h)`.