Your LAST Name _____ **Your FIRST Name** _____

Your Collaborators

- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This homework covers:

- CTFT
- Modulation

HW10.1 (A Few More Salient Signals & Their CTFTs) Determine X, the continuous-time Fourier transform (CTFT) of each of the following signals x.

(a) **Signum:** Let the signal x be characterized by $x(t) = \text{sgn}(t), \forall t$, where

$$\text{sgn}(t) = \begin{cases} +1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$

is, according to one popular definition, the *signum function.*

Hint: Write the signum function in terms of the unit step function and a constant. Then use the differentiation property of the CTFT to arrive at the answer.

(b) **Unit Step:** $x(t) = u(t), \ \forall t$.

Hint: Write the unit step $u(t)$ in terms of the signum function $\text{sgn}(t)$ and a constant. Then use your result from the previous part to determine the CTFT.

(c) **Infinite Impulse Train:** $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \forall t$, where $0 < T$.

Hint: This signal is periodic; it has a Fourier series (FS) expansion. What is the transform of each term in the FS expansion of x? How does the shape of the CTFT of x change as T varies between 0 to $+\infty$?

(d) **Rectangular pulse:**

$$x(t) = \begin{cases} b & |t| \leq a \\ 0 & \text{elsewhere.} \end{cases}$$

The parameters a and b have appropriately-chosen positive values.

HW10.2 A Close and Normal Family Find the CTFT $X(\omega)$ of $x(t)$, where

$$\forall t \in \mathbb{R},\ x(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

and $\sigma > 0$. You may or may not find it useful to know that

(i) $tg(t) \overset{\mathcal{F}}{\leftrightarrow} i\frac{\mathrm{d}G(\omega)}{\mathrm{d}\omega}$

(ii) $\int_{-\infty}^{+\infty} x(t)dt = 1$

(iii) $\int_a^b u\,\mathrm{d}v = uv\Big|_a^b - \int_a^b v\,\mathrm{d}u$

(iv) $\int_0^\omega \frac{1}{G(\lambda)}\frac{\mathrm{d}G(\lambda)}{\mathrm{d}\lambda}d\lambda = \ln G(\omega)\Big|_0^\omega$

Hint: Take the derivative of the given equation for $x(t)$ and use the CTFT properties and the above hints to take the Fourier transform of both sides.

HW10.3 CTFT for Heat Flow In 1822, Joseph Fourier published *Théorie analytique de la chaleur (The Analytical Theory of Heat)*, in which he developed and solved the heat equation, laying the groundwork for the same Fourier Transform we use today.

Suppose we have a very long rod (one that is essentially infinite in length, so that boundary conditions can be ignored). Let the temperature at position $x \in \mathbb{R}$ along the rod at time $t \geq 0$ be given by $u(x,t)$. The heat equation says that the temperature changes according to

$$\frac{\partial}{\partial t}u(x,t) = \alpha \frac{\partial^2}{\partial x^2}u(x,t)$$

where $\alpha > 0$ is a constant that depends on the thermal diffusivity of the rod's material. As an initial condition, the temperature along any point of the rod at time zero is given as $u(x,0) = g(x)$.

In this problem, we will use the Fourier Transform to solve the heat equation. However, we will be taking the Fourier Transform with respect to the spatial variable x rather than the time variable t as we typically do. To this end, define

$$U(\omega,t) = \int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx$$

so that

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{i\omega x}d\omega$$

Note that instead of working back and forth between the time and temporal frequency domains as we usually do, we're now working between the space and spatial frequency domains. Fortunately, all the same principles and properties you're familiar with carry over.

(a) Show that $\frac{\partial}{\partial t}U(\omega,t) = -\alpha\omega^2 U(\omega,t)$.

Hint: Remember that when taking a partial derivative with respect to a variable, all other variables are treated as constants.

(b) Show that

$$\frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}U(\omega,t)\right] = 0$$

(c) Integrate both sides of the result of the previous part with respect to t, and then apply the initial condition given to show that

$$U(\omega, t) = G(\omega)e^{-\alpha\omega^2 t}$$

where $G(\omega)$ is the spatial Fourier Transform of $g(x) = u(x, 0)$. Remember that a function constant in x will have a derivative of 0 with respect to t.

(d) Express $u(x, t)$ as a convolution integral in terms of the signal g. You may find the Fourier Transform pair

$$\frac{1}{2\sqrt{\pi\kappa}}e^{-x^2/4\kappa} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\kappa\omega^2}$$

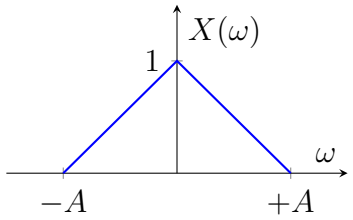
to be useful. Here, $\kappa > 0$ is some real constant, and x is the variable over which the transform is taken.

(e) Given the initial condition $u(x,0) = \delta(x)$, solve for $u(x,t)$ using your result in the previous part.[1]

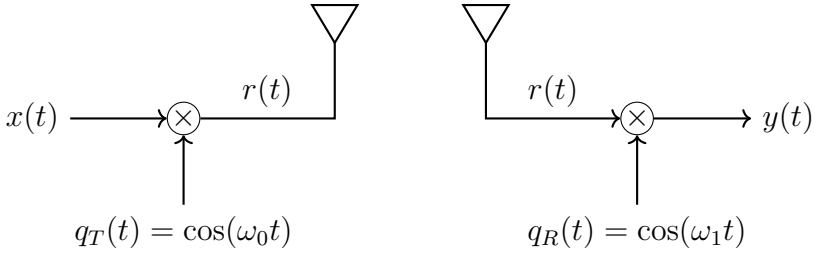
(f) Determine an expression for, and provide well-labelled plots of, $u(x, t = \frac{1}{2\alpha})$ and $u(x, t = \frac{3}{2\alpha})$ on the same graph. What happens to the spatial distribution of the heat in the rod as time goes on?

[1] If you like, you can think of the integral of the rod's temperature over all space, $\int_{-\infty}^{\infty} u(x,t)dx$, as representing the total amount of heat in it at time t. This isn't quite correct due to units not working out, but it's a passable analogy for this problem. Using this metaphor, the initial condition $g(x) = \delta(x)$ means that we have one unit of heat placed in the middle of the rod at time $t = 0$, and we want to see how it spreads out.

HW10.4 (Frequency Mismatch in Amplitude Modulation) A bandlimited continuous-time signal x has the triangular spectrum shown below:



The following diagram shows an amplitude modulation-demodulation scheme to communicate the signal x to a receiver. In this problem, you'll explore the effects



of frequency mismatch between the transmitter and receiver carriers q_T and q_R, respectively. In particular, assume that

$$0 < \epsilon \ll A < \omega_0 = \omega_1 + \epsilon$$

where ϵ is unknown.

(a) Determine reasonably simple expressions for the signals r and y. You may find the following trigonometric identity useful:

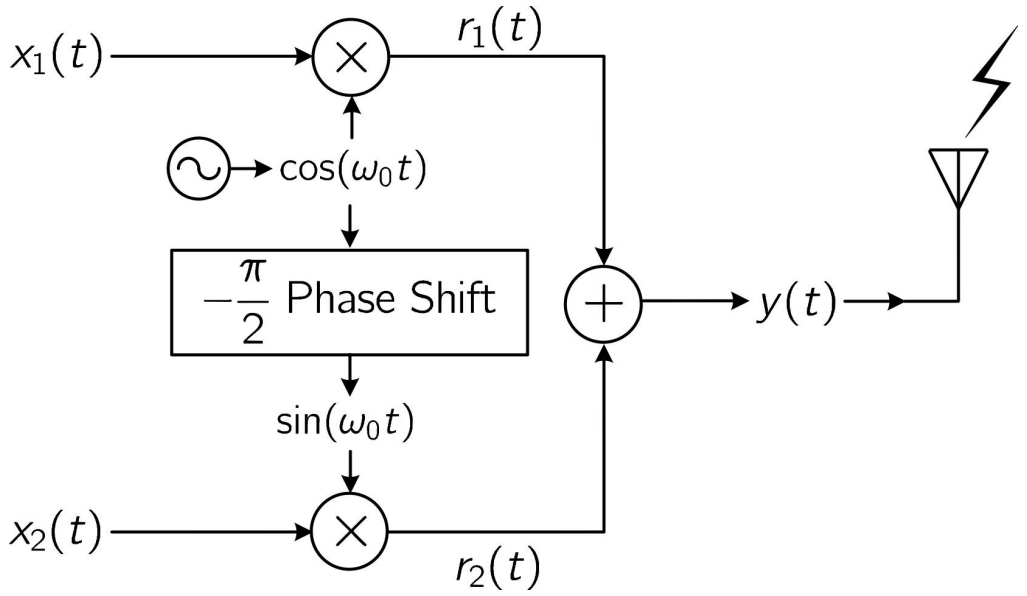
$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

(b) Provide well-labeled plots of $R(\omega)$ and $Y(\omega)$, the spectra of the signals r and y, respectively.

(c) Explain why the information-bearing signal x is irrecoverable, even if we send the signal y through an ideal low-pass filter.

Hint: Recall that ϵ is unknown.

HW10.5 (Quadrature Multiplexing)
The figure below illustrates a transmission scheme known as *quadrature multiplexing*, by which we can embed two bandlimited continuous-time signals on the *same* carrier frequency. This method is also referred to as *quadrature-carrier multiplexing* and *quadrature amplitude modulation (QAM)*. It allows the information-bearing signals to have overlapping spectra—that is, footprints that compete for real estate along the frequency axis.



The trigonometric identities below may be useful to you in one or more parts of this problem:

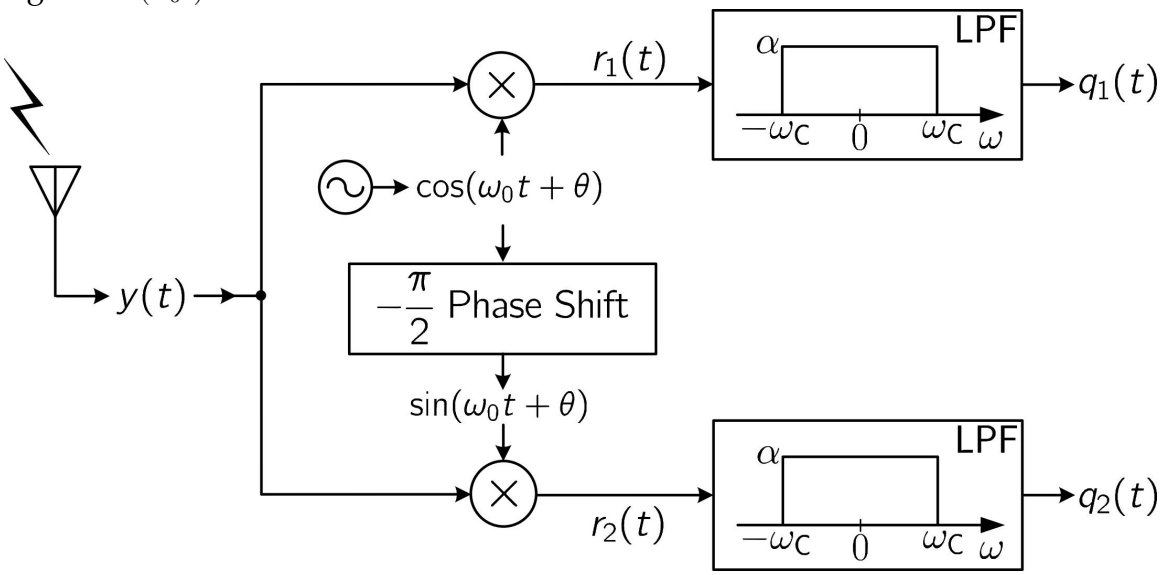
$$\begin{aligned} \cos a \, \cos b &= \frac{1}{2}\Big[\cos(a+b) + \cos(a-b)\Big] \\ \sin a \, \cos b &= \frac{1}{2}\Big[\sin(a+b) + \sin(a-b)\Big]. \end{aligned}$$

(a) Show that the transmitted signal y can be expressed in the amplitude-phase form

$$\forall t \in \mathbb{R}, \quad y(t) = A(t)\ \cos\big(\omega_0 t + \phi(t)\big),$$

and determine how each of the signals x_1 and x_2 contributes to the amplitude A and the phase ϕ of the transmitted signal y. In other words, express each of $A(t)$ and $\phi(t)$ in terms of $x_1(t)$ and $x_2(t)$.

The figure below shows a receiver demodulator for the QAM transmission scheme described above. Note that the local oscillator at the receiver has a phase error of θ (radians) with respect to its transmitter counterpart, which produces the carrier signal $\cos(\omega_0 t)$.



The passband gain of each ideal low-pass filter is $\alpha = 2$, and its cutoff frequency is much smaller than the carrier frequency, yet sufficient to pass the baseband content of interest; that is, $A < \omega_c \ll \omega_0$.

(b) Determine a reasonably simple expression for the output signals q_1 and q_2. Your answers should be in terms of the information-bearing signals x_1 and x_2, and the phase error θ. *You may continue on the next page.*

More space for part (b)

(c) In this part, assume that the phase error is very small ($|\theta| \ll 1$).

(i) Simplify your expressions for q_1 and q_2 in part (b) based on the assumption that the phase error is very small.
Use the small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.

(ii) Suppose the information-bearing signals x_1 and x_2 are of equal magnitudes at each time instance: $|x_1(t)| = |x_2(t)|$, for all t. What's the largest allowable phase error θ such that the interference of x_2 in q_1 (or, equivalently, the interference of x_1 in q_2) does not exceed 1% (i.e., -40 dB)? Express your answer in radians and degrees.