Problem Set 8
Due: Fri, 23 Oct 2026, 23:59
Accepted Thru: Sun, 25 Oct 2026, 23:59

Your LAST Name _____ **Your FIRST Name** _____

Your Collaborators

- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This homework covers:

- DTFS
- DFT

HW8.1 (DTFS Expansion) Determine the complex-exponential discrete-time Fourier series (DTFS) expansion for each signal $x : \mathbb{Z} \to \mathbb{R}$ described below, or explain why no such expansion exists. For each case where a DTFS expansion exists, be sure to identify the period p and the fundamental frequency ω_0.

For parts (b) and (c), assume x denotes *exactly one period* of a periodic signal $\widetilde{x}$. Your answer would then be the DTFS expansion of the periodic signal $\widetilde{x}$ for certain values of n, and zero otherwise.

(a) $x[n] = \sum_{l=-\infty}^{+\infty} \delta[n - lp]$, where p is a positive integer.

(b) $x[n] = \delta[n+2] + 2\delta[n+1] + 3\,\delta[n] + 2\,\delta[n-1] + \delta[n-2]$.

(c) $x[n] = \delta[n+2] + \delta[n+1] - \delta[n-1] - \delta[n-2]$.

HW8.2 (LTI Processing of Periodic Signals) Consider a periodic, discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the discrete-time Fourier series (DTFS) expansion

$$x[n] = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \,,$$

where ω_0 denotes the fundamental frequency of the signal; if p is the period of x, then $\omega_0 = 2\pi/p$.

Suppose x is the input signal applied to a linear time-invariant (LTI) system characterized by the impulse response $h : \mathbb{Z} \to \mathbb{R}$ and corresponding frequency response H, where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] \, e^{-i\omega n} \,, \quad \forall \omega.$$

Let y be the corresponding output signal.

(a) Let the DTFS expansion of the output signal y be

$$y[n] = \sum_{k=\langle p \rangle} Y_k \, e^{ik\omega_0 n} \,.$$

(i) Express the output-signal DTFS coefficients Y_k in terms of the input signal DTFS X_k and the frequency response H.

(ii) Suppose the impulse response of the LTI system is given by $h[n] = \delta[n - n_0]$, where $n_0 \in \mathbb{Z}$. Explicitly determine the output-signal DTFS coefficients Y_k in terms of the input-signal DTFS coefficients X_k.

HW8.3 (A DTFS Puzzle) Consider a discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the following the properties:

- $x[n + 4l] = x[n],\ \forall l, n \in \mathbb{Z}$.
- $\displaystyle\sum_{n=-1}^{2} x[n] = 4$.
- $\displaystyle\sum_{n=-1}^{2} (-1)^n x[n] = 8$.
- $\displaystyle\sum_{n=-1}^{2} x[n] \cos\left(\frac{\pi}{2}n\right) = \sum_{n=-1}^{2} x[n] \sin\left(\frac{\pi}{2}n\right) = 0$.

(a) Determine the complex exponential Fourier series coefficients X_{-1}, X_0, X_1, and X_2 for the signal x. From the coefficients, determine and provide a well-labeled plot for the signal x.

More space for part (a):

(b) Based on your results from part (a), determine the fundamental period p and fundamental frequency ω_0 of the signal x. Express x in terms of its DTFS coefficients and complex exponentials of appropriate frequencies, and identify each coefficient and its corresponding harmonic frequency.

HW8.4 (DTFS as a Change of Basis) In this problem, we will build intuition on Fourier transforms by looking at the DTFS through the lens of change of basis.

We chose the DTFS for this problem because discrete, periodic signals lend themselves well to the vector operations many of you have seen in basic linear algebra.

If we look at the space of discrete-time signals with period p, we can represent each signal as a vector $\in \mathbb{R}^p$ consisting of one period of the signal. To be specific, given a signal $x[n]$ with period p, we can write it as a vector

$$\boldsymbol{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix}$$

(a) To familiarize yourself with this vector notation, find the fundamental period p of $x[n] = \cos(\pi n) + \sin\left(\frac{\pi}{2}n\right)$ and write $x[n]$ as a vector, $\boldsymbol{x}$, in $\mathbb{R}^p$.

(b) We would like to define an orthogonal basis of complex exponentials for $\mathbb{R}^p$ by using the basis signals

$$\psi_k[n] = e^{i\omega_k n}$$

Remember, we are working in the space of p-periodic functions, so $\psi_k[n]$ must correspond to a complex exponential with period p.

Show, for $\omega_k = k\frac{2\pi}{p}$, $\psi_k[n]$ will have a period of p. Explain why only p unique $\boldsymbol{\psi}_k$ exist.

Hint: Think about the roots of unity.

(c) Show that the set vectors $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ are orthogonal to each other. A set of vectors $\{\boldsymbol{v}_k\}$ in $\mathbb{R}^p$ is defined as being orthogonal if $\boldsymbol{v}_l^T \boldsymbol{v}_m^* = 0, \forall l \neq m$, where the inner product of two complex-valued vectors is defined as follows:

$$\boldsymbol{a}^T \boldsymbol{b}^* = \sum_{n=0}^{p-1} a[n] b^*[n],$$

where $\boldsymbol{b}^*$ is the complex conjugate of $\boldsymbol{b}$ and $b^*[n]$ is the complex conjugate of $b[n]$.

(d) What is the norm of $\boldsymbol{\psi}_k$?

(e) Because the set of vectors $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ are orthogonal to each other and there are p of them, we can conclude $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ is an orthogonal basis for $\mathbb{R}^p$. Therefore, we can write out any vector $\boldsymbol{x} \in \mathbb{R}^p$ as a linear combination of projections of $\boldsymbol{x}$ onto the vectors in $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$.

Recall

$$\boldsymbol{x} = \sum_k \text{proj}_{\boldsymbol{\psi}_k} \boldsymbol{x} \tag{1}$$

where

$$\text{proj}_{\boldsymbol{\psi}_k} \boldsymbol{x} = \frac{\boldsymbol{x}^T \boldsymbol{\psi}_k^*}{||\boldsymbol{\psi}_k||^2} \boldsymbol{\psi}_k.$$

Show that

$$\text{proj}_{\boldsymbol{\psi}_k} \boldsymbol{x} = X_k \boldsymbol{\psi}_k,$$

where X_k is the k^{th} DTFS coefficient of $x[n]$.

Hint: Use the DTFS analysis equation.

(f) Show that equation (1) is the DTFS synthesis equation.

(g) We now stack X_k into the column vector $\boldsymbol{X} = \begin{bmatrix} X_0 & X_1 & \cdots & X_{p-1} \end{bmatrix}^T$. Using the DTFS representation of $x[n]$, derive the matrix form of the DTFS synthesis equation, also known as the synthesis equation. That is, find $\boldsymbol{\Psi}$, where

$$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{X}$$

You do not have to simplify the complex exponentials in $\boldsymbol{\Psi}$.

(h) Is $\mathbf{\Psi}$ from part (g) an orthogonal matrix? Is it an orthonormal matrix? Why or why not?

Hint: Try to express $\mathbf{\Psi}$ in terms of the vectors $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$, then use parts (c) and (d).

(i) Using the previous two parts, derive **Parseval's Theorem**:

$$\frac{1}{p}\sum_{n=0}^{p-1}|x[n]|^2 = \sum_{k=0}^{p-1}|X_k|^2 \iff \frac{1}{p}||\boldsymbol{x}||^2 = ||\boldsymbol{X}||^2$$

This relationship directly correlates the "energy" located in the time domain to the energy located in the frequency domain.

Hint: If matrix $\mathbf{A}$ is orthonormal, then $||\mathbf{A}\boldsymbol{x}|| = ||\boldsymbol{x}||$. Is $\mathbf{\Psi}$ orthonormal? If not, can we make it orthonormal?

HW8.5 (The DTFS and Its Fraternal Twin, the DFT) To represent finite-length or periodic discrete-time signals, engineers have traditionally used a complex exponential Fourier series expansion that is slightly different from the Discrete-Time Fourier Series (DTFS). The expansion of choice is called the Discrete Fourier Transform (DFT).[1]

In this problem, you will discover the simple relationship between the DTFS and the DFT. Recall that the analysis and synthesis equations of the DTFS expansion of a periodic discrete-time signal $x : \mathbb{Z} \to \mathbb{C}$ are:

$$x[n] \;=\; \sum_{k=\langle p \rangle} X_k \, \psi_k[n] \qquad \overset{\text{DTFS}}{\longleftrightarrow} \qquad X_k \;=\; \frac{\langle x, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle}.$$

Written in the time domain, these equations become:

$$x[n] \;=\; \sum_{k=\langle p \rangle} X_k \, \psi_k[n] \qquad \overset{\text{DTFS}}{\longleftrightarrow} \qquad X_k \;=\; \frac{1}{p} \sum_{n=\langle p \rangle} x[n] \, \psi_k^*[n],$$

where $\psi_k[n] \;=\; e^{\,i\,k\,\omega_0\,n}$, $\quad \langle x, \psi_k \rangle \;=\; \sum_{n=\langle p \rangle} x[n] \; \psi_k^*[n]$, $\quad \langle \psi_k, \psi_k \rangle \;=\; \sum_{n=\langle p \rangle} \psi_k[n] \, \psi_k^*[n]$

(a) Consider the set of basis signals $\Phi = \{\phi_0, \ldots, \phi_{p-1}\}$, where $\phi_k = \frac{1}{p}\,\psi_k$. Determine $\langle \phi_k, \phi_l \rangle$ for the cases $k = l$ and $k \neq l$. Use what you already know about $\langle \psi_k, \psi_l \rangle$ to simplify your work.

[1] The Fast Fourier Transform (FFT) is merely a reference to a family of computationally-efficient algorithms for implementing the DFT. The FFT is not a different transform.

(b) We wish to express a p-periodic signal x in terms of the basis functions in Φ, as follows:

$$x[n] \;=\; \sum_{k=\langle p \rangle} X[k]\; \phi_k[n].$$

Determine an expression for the coefficients $X[k]$ in terms of projections of the signal x onto the basis functions in ϕ. What is the relationship between the DFT coefficients $X[k]$ and the DTFS coefficients X_k? Finally, write down the analysis and synthesis equations for the DFT.

(c) Define $\mathbf{x}$ and $\mathbf{X}_{DFT}$ to be the signal and DFT coefficient vectors, respectively:

$$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix}, \qquad \mathbf{X}_{DFT} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[p-1] \end{bmatrix}.$$

Express the analysis and synthesis equations you found in (b) as a matrix-vector formulation. In other words, find $\mathbf{F}_{DFT}$ and $\mathbf{F}_{iDFT}$ such that

$$\mathbf{X}_{DFT} = \mathbf{F}_{DFT}\,\mathbf{x}, \qquad \mathbf{x} = \mathbf{F}_{iDFT}\,\mathbf{X}_{DFT}.$$

Express $\mathbf{F}_{DFT}$ and $\mathbf{F}_{iDFT}$ in terms of the complex exponential.