Problem Set 9
Due: Fri, 6 Nov 2026, 23:59
Accepted Thru: Sun, 8 Nov 2026, 23:59

Your LAST Name _____ **Your FIRST Name** _____

Your Collaborators

- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This homework covers:

- CTFS
- DTFT

HW9.1 (A CTFS Puzzle) FourierBot knows the following clues about a periodic, continuous-time signal $x(t)$ and its CTFS coefficients X_k:

(i) $x(t) = x(t+4)$ for all $t \in \mathbb{R}$.

(ii) For $0 \leq k < 4$, there exists $t \in \mathbb{R}$ such that $x(t) \neq x(t+k)$.

(iii) $X_k = X_{-k}^*$ for all $k \in \mathbb{N}$.

(iv) $X_k = 0$ for $k > 2$.

(v) The average value of $x(t)$ over one period is 4.

(vi) $\int_{-3}^{1} |x(t)|^2 dt = 128$.

(vii) $x(t)$ is even.

(viii) $x(0) = 12$.

Determine their signal $x(t)$ **in terms of sines and cosines** (with a potential constant term).

HW9.2 (Continuous-Time Trigonometric Fourier Series)
A periodic, real-valued continuous-time signal x can be represented by the trigonometric Fourier series expansion

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{+\infty} A_k \cos \frac{2\pi k}{p} t + \sum_{\ell=1}^{+\infty} B_\ell \sin \frac{2\pi \ell}{p} t \\ &= A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{\ell=1}^{+\infty} B_\ell \sin \ell\omega_0 t \,, \end{aligned} \tag{1}$$

where $\omega_0 = \frac{2\pi}{p}$ is the fundamental frequency, and p the fundamental period, of x. Equation 1 is the *synthesis equation* of the trigonometric Fourier series expansion.

More compactly,

$$x = \sum_{k=0}^{+\infty} A_k \chi_k + \sum_{\ell=1}^{+\infty} B_\ell \phi_\ell,$$

which is a linear combination of functions χ_k and ϕ_ℓ, where

$$\begin{aligned} \chi_k(t) &= \cos k\omega_0 t, \quad k \in \mathbb{Z}^+, \text{and} \\ \phi_\ell(t) &= \sin \ell\omega_0 t, \quad \ell \in \mathbb{N}. \end{aligned}$$

Recall that $\mathbb{Z}^+ \triangleq \{0, 1, 2, \ldots\}$ and $\mathbb{N} \triangleq \{1, 2, 3, \ldots\}$.

In this problem, you will show that

$$\{\chi_0, \chi_1, \phi_1, \chi_2, \phi_2, \chi_3, \phi_3, \ldots\}$$

is a set of mutually-orthogonal functions. You will then exploit the mutual orthogonality of these functions to determine the coefficients A_k and B_ℓ.

(a) In what follows you're given a set of inner products, and you're asked to determine a fairly simple expression for each. You may proceed in one of two ways.

In one method, you can express $\chi_k(t)$ and $\phi_\ell(t)$ in terms of the complex exponential functions

$$\psi_r(t) = e^{ir\omega_0 t}, \quad r \in \mathbb{Z},$$

and exploit the orthogonality property of the complex exponentials to arrive at the appropriate expression for each inner product that you're asked to characterize.

Alternatively, you can find each of the inner products in (i)–(iv) below by inserting the appropriate functions in the definition of the inner product. If you choose this method, the following trigonometric identities may prove helpful to you:

$$\begin{aligned} \cos\alpha \cos\beta &= \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta) \\ \cos\alpha \sin\beta &= \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta) \end{aligned}$$

(i) $\langle \chi_0, \chi_0 \rangle$.

(ii) $\langle \chi_k, \chi_k \rangle$, where $k \neq 0$.

(iii) $\langle \chi_k, \chi_m \rangle$, where $k \neq m$.

(iv) $\langle \chi_k, \phi_\ell \rangle$, where $k \in \mathbb{Z}^+$ and $\ell \in \mathbb{N}$.

(b) Take the inner product of each side of Equation 1 with an appropriate function χ_k or ϕ_ℓ, and exploit the mutual orthogonality results you obtained in part (a) to determine the following expressions for the coefficients A_k and B_ℓ, where $k \in \mathbb{Z}^+$ and $\ell \in \mathbb{N}$. These are the *analysis equations* of the trigonometric Fourier series.

$$\begin{aligned} A_0 &= \frac{1}{p}\int_{\langle p\rangle} x(t)dt. \\ A_k &= \frac{2}{p}\int_{\langle p\rangle} x(t)\cos k\omega_0 t\, dt, \quad 1 \leq k. \\ B_\ell &= \frac{2}{p}\int_{\langle p\rangle} x(t)\sin \ell\omega_0 t\, dt, \quad 1 \leq \ell. \end{aligned}$$

(c) Recall that the complex exponential Fourier series expansion of x is

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k \, e^{ik\omega_0 t},$$

and exploit the fact that x is real-valued (which means that the coefficients X_k are conjugate-symmetric, i.e., $X_k^* = X_{-k}$), to establish the following relations between the trigonometric and complex exponential Fourier series coefficients:

$$\begin{aligned} A_0 &= X_0 \\ A_k &= +2\,\mathsf{Re}(X_k), \quad 1 \leq k \\ B_\ell &= -2\,\mathsf{Im}(X_\ell), \quad 1 \leq \ell. \end{aligned}$$

HW9.3 DTFT Warmup Determine the DTFT or inverse DTFT for the following subproblems.

(a)

$$x(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

(b)

$$x[n] = \begin{cases} n & \text{if } |n| \leq 3 \\ 0 & \text{if } |n| > 3 \end{cases}$$

(c)

$$X(\omega) = \sum_{k=-\infty}^{\infty} (-1)^k \delta\left(\omega - \frac{\pi}{2}k\right)$$

HW9.4 (DTFT Property Practice) Note that we can break a signal into even and odd signals. For any $x[n]$, we can find its even and odd components with the following equations,

$$x_e[n] = \frac{x[n] + x[-n]}{2} \qquad x_o[n] = \frac{x[n] - x[-n]}{2} \qquad x[n] = x_e[n] + x_o[n]$$

Note that odd signals are zero at timestep zero, so $x_o[0] = 0$ and $x[0] = x_e[0]$.

By linearity, $x_e[n] \xleftrightarrow{\mathcal{F}} X_e(\omega) = \frac{X(\omega)+X(-\omega)}{2}$.

It also can be seen that $\text{Re}\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2}$.

For this problem, consider a ***real-valued***, *causal* discrete-time signal x and its Fourier transform X. The real part of the signal's Fourier transform is given by

$$\text{Re}\{X(\omega)\} = 1 + \cos(\omega) - \cos(2\omega).$$

(a) Using DTFT properties, evaluate the integral $\int_{\langle 2\pi \rangle} X(\omega)\, d\omega$.

Hint: How are $X_e(\omega)$ and $\text{Re}\{X(\omega)\}$ related for a real-valued signal?

(b) Determine, and provide a well-labeled plot of, the signal x.

(c) Evaluate the integral $\int_{\langle 2\pi \rangle} X(\omega)\ \cos(\omega)\, d\omega$.

(d) Evaluate the integral $\int_{\langle 2\pi \rangle} |X(\omega)|^2\, d\omega$.

(e) Determine a reasonably simple expression for $\text{Im}\{X(\omega)\}$, the imaginary part of the signal's Fourier transform, where, as you know,

$$X(\omega) = \text{Re}\{X(\omega)\} + i\,\text{Im}\{X(\omega)\}.$$

HW9.5 (Reversing the DTFT) Consider a ***real-valued***, *causal*[1] discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having sample values $x[n]$. The discrete-time Fourier transform (DTFT) of x is given by the function $X : \mathbb{R} \to \mathbb{C}$ whose values are denoted by $X(\omega)$.

The following parts are independent. Information given in one part should not be assumed to hold for the others.

(a) The magnitude-squared of X, denoted by $|X(\omega)|^2$, is given by

$$|X(w)|^2 = \frac{5}{4} - \cos\omega \ .$$

Determine a choice for the signal x. Is your answer unique? If so, explain why. If not, find multiple signal that satisfies the property given above.

Hint: Given a complex-valued function $f(x)$, we know that $|f(x)|^2 = f(x)f^*(x)$.
Hint: Think about how to factor the above expression. Start this problem by converting $\cos\omega$ into complex exponentials and distributing each exponential to each different term.

[1]Technically, it is an abuse of terminology to call a signal "causal"; causality is defined for a *system*, not a *signal*. Recall that the impulse response h of a causal LTI system satisfies the constraint, $h : \mathbb{Z} \to \mathbb{C}, h[n] = 0, \forall n < 0$ (if h corresponds to a discrete-time system), or $h : \mathbb{R} \to \mathbb{C}, h(t) = 0, \forall t < 0$ (if h corresponds to a continuous-time system). Borrowing from this LTI vernacular, we say that a discrete-time signal $x : \mathbb{Z} \to \mathbb{C}$ is causal if $x[n] = 0, \forall n < 0$ and a continuous-time signal $x : \mathbb{R} \to \mathbb{C}$ is causal if $x(t) = 0, \forall t < 0$.

(b) The imaginary part of $X(\omega)$, denoted by $\text{Im}\{X(\omega)\}$, is characterized by

$$\text{Im}\{X(w)\} = -\sin(2\omega)\,,$$

and it is further known that

$$X(\omega)|_{\omega=0} = 1\ .$$

Determine the signal x. Is your answer unique? If so, explain why. If not, find another signal that satisfies the properties given above.

Hint: It can be seen that $\text{Im}\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2i}$.