Your LAST Name ________________ **Your FIRST Name** ________________

Your Collaborators

- Name: ____________________________ SID: ______________________
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Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This problem set is designed to strengthen your mathematical foundations. In particular, you will get practice with complex numbers and their arithmetic and complex variables and their algebra; you will also tiptoe into the realm of complex-valued functions (with more to come later).

Getting comfortable with the geometry of the complex plane and developing fluency with using a graphical approach to solve problems that involve complex numbers, variables, and functions should be among your main achievements this week.

Reading

Our course has *no* official textbook; however, as a prelude to this problem set, we recommend that you read Appendix B of the book *Structure and Interpretation of Signals and Systems* by Edward Lee & Pravin Varaiya (http://leevaraiya.com/).

The authors—emeriti faculty of our EECS Department—provided the download link to a free PDF file of their book. Roots of unity, a topic central to this assignment, is also mentioned in Example B.2.

HW1.1 Two complex numbers z_1 and z_2 are described below:

$$z_1 = 1 + i\sqrt{3} \qquad\qquad z_2 = \exp\left(i\,\frac{2\pi}{3}\right)\ .$$

Throughout this problem, express each of your answers in Cartesian form $(a+ib)$, in polar form ($re^{i\theta}$, where $r > 0$), as a real number, as an imaginary number, or graphically in a well-labeled complex-plane diagram.

(a) Identify each of the following complex numbers as points (or vectors) on the complex plane, using a well-labeled sketch: z_1, z_2, z_1^*, z_2^*, $1/z_1$, and $1/z_2$.

$z_1 = 1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\exp\left(i\frac{\pi}{3}\right)$

$z_2 = \exp\left(i\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

See Figure 1.

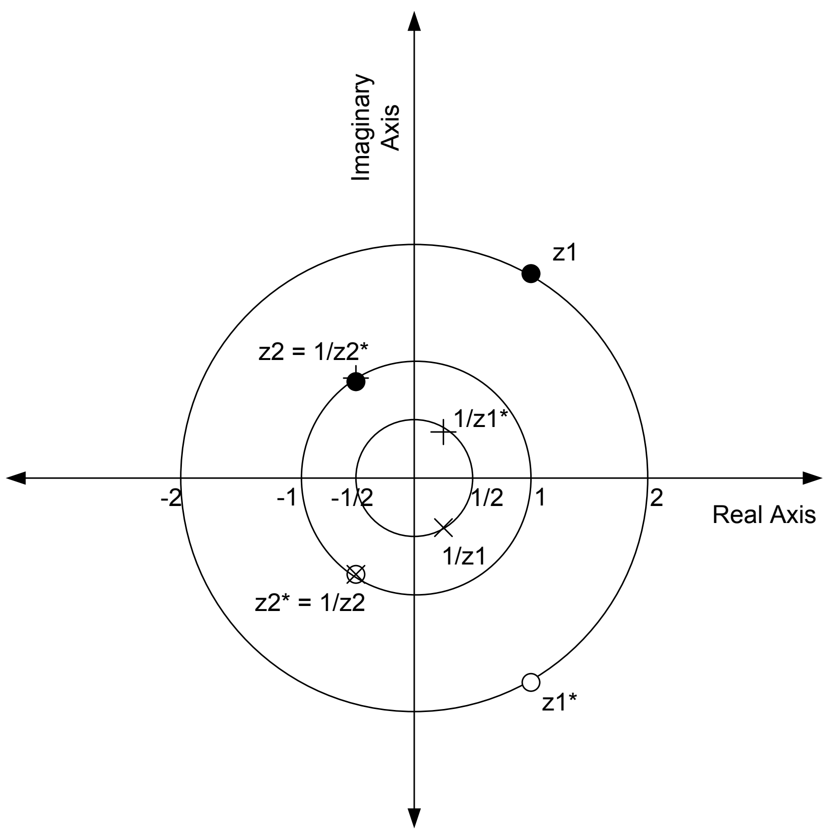


Figure 1: Problem 1.2a

(b) Determine a reasonably simple expression or value for each of the following powers of z_1 and z_2:

(i) z_1^2

$z_1^2 = -2 + i2\sqrt{3} = 4\exp\left(i\frac{2\pi}{3}\right) = 4z_2$ (all answers reasonable)

(ii) z_1^3

$z_1^3 = \left(2\exp\left(i\frac{\pi}{3}\right)\right)^3 = 8\exp\left(i\frac{3\pi}{3}\right) = -8$

(iii) z_1^6

$z_1^6 = \left(2\exp\left(i\frac{\pi}{3}\right)\right)^6 = 64\exp\left(i\frac{6\pi}{3}\right) = 64\exp\left(i2\pi\right) = 64$

(iv) z_2^4

$z_2^4 = \left(\exp\left(i\frac{2\pi}{3}\right)\right)^4 = \exp\left(i\frac{8\pi}{3}\right) = \exp\left(i\left(2\pi + \frac{2\pi}{3}\right)\right) = \exp\left(i\frac{2\pi}{3}\right) = z_2$

HW1.2 Consider a pair of complex numbers $z = a + bi$ and $v = c + di$, where $a, b, c, d \in \mathcal{R}$. Also let $z = r_1 e^{i\theta_1}$, and $v = r_2 e^{i\theta_2}$ for appropriate nonnegative values of r_1, r_2 and appropriate real values for θ_1, θ_2. Prove the following statements.

(a) $z + z^* = 2a$.

$z + z^* = (a + bi) + (a - bi) = 2a.$

(b) $z\, z^* \geq 0$, with equality if, and only if, $z = 0$.

Positivity (recall $a, b, \in \mathbb{R}$):
$zz^* = (a + ib)(a - ib) = a^2 + b^2 \rightarrow a^2 + b^2 \geq 0$
Equality if and only if $z = 0$:
$zz^* = r_1 e^{i\theta_1} r_1 e^{-i\theta_1} = r_1^2 e^{i\theta_1 - i\theta_1} = r_1^2 = 0 \iff r_1^2 = 0 \iff r_1 = 0 \iff z = 0$

(c) z is real if, and only if, $z = z^*$.

$z \in \mathbb{R} \iff z = a + 0i = a - 0i = z^*$

(d) $(z\, v)^* = z^*\, v^*$.

$(zv)^* = \left(r_1 e^{i\theta_1} r_2 e^{\theta_2}\right)^* = \left(r_1 r_2 e^{i(\theta_1 + \theta_2)}\right)^* = r_1 r_2 e^{-i(\theta_1 + \theta_2)} = r_1 e^{-i\theta_1} r_2 e^{-i\theta_2} = z^* v^*$

HW1.3 Consider the following sixth-order equation:

$$z^6 - 2\sqrt{3}z^4 + 4z^2 = 0.$$

Determine the six solutions (roots) of the equation, and express each root in a simple rectangular form. Explain your work succinctly, but clearly and convincingly.

Since the polynomial has powers z^2, z^4, z^6, let us first factor this in terms of z^2:

$$\begin{aligned} z^6 - 2\sqrt{3}z^4 + 4z^2 =& z^2\left(z^4 - 2\sqrt{3}z^2 + 4\right) \\ =& z^2\left[z^2 - (\sqrt{3}+i)\right]\left[z^2 - (\sqrt{3}-i)\right] = 0 \end{aligned}$$

The last step comes from the quadratic equation. For the product to be zero, at least one of the factors must be zero. Let us take each term in turn:

$$z^2 = 0 \implies z = 0 \text{ (double root)}$$

$$\begin{aligned} z^2 - (\sqrt{3}+i) =& 0 \\ z^2 =& \sqrt{3}+i = 2e^{i\pi/6} \\ \Rightarrow z =& \sqrt{2}e^{i\pi/12}, \sqrt{2}e^{i13\pi/12} \end{aligned}$$

$$\begin{aligned} z^2 - (\sqrt{3}-i) =& 0 \\ z^2 =& \sqrt{3}-i = 2e^{-i\pi/6} \\ \implies z =& \sqrt{2}e^{-i\pi/12}, \sqrt{2}e^{i11\pi/12} \end{aligned}$$

The six roots are: $0, 0, \sqrt{2}e^{i\pi/12}, \sqrt{2}e^{i13\pi/12}, \sqrt{2}e^{-i\pi/12}, \sqrt{2}e^{i11\pi/12}$ (see Figure 2).

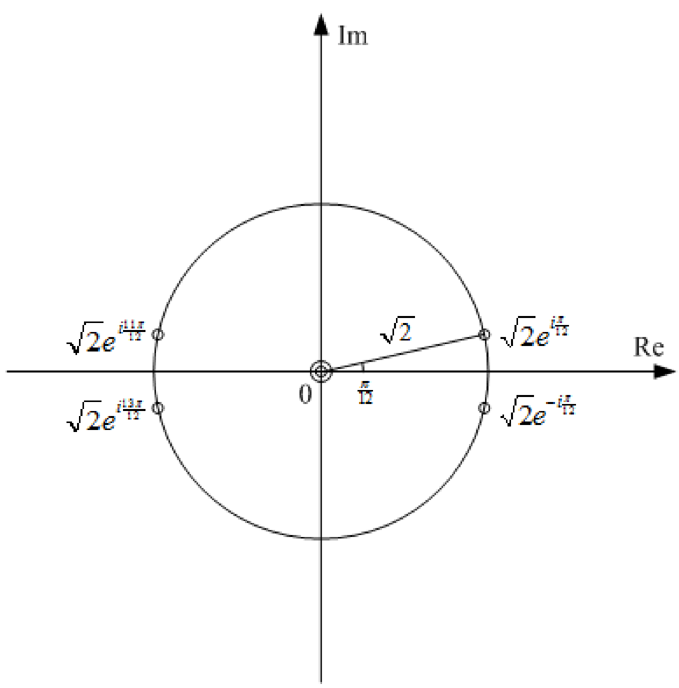


Figure 2: Problem 1.3.

Converting these roots to rectangular form, we have: $(0,0)$, $(0,0)$, $(\sqrt{2}\cos\frac{\pi}{12}, \sqrt{2}\sin\frac{\pi}{12})$, $(\sqrt{2}\cos\frac{13\pi}{12}, \sqrt{2}\sin\frac{13\pi}{12})$, $(\sqrt{2}\cos\frac{-\pi}{12}, \sqrt{2}\sin\frac{-\pi}{12})$, $(\sqrt{2}\cos\frac{11\pi}{12}, \sqrt{2}\sin\frac{11\pi}{12})$.

HW1.4 Consider a complex number $z = e^{i\theta}$.

(a) Show that

$$\sum_{n=0}^{N} z^n = \begin{cases} N+1 & \text{if } \theta = 0 \\ \dfrac{\sin\left[\dfrac{(N+1)\theta}{2}\right]}{\sin\dfrac{\theta}{2}} \exp\left(i\dfrac{N\theta}{2}\right) & \text{if } \theta \neq 0. \end{cases}$$

For $\theta = 0$ we have:

$$\sum_{n=0}^{N} z^n = \sum_{n=0}^{N} 1 = N+1$$

For $\theta \neq 0$:

$$\sum_{n=0}^{N} z^n = \sum_{n=0}^{N} \left(e^{i\theta}\right)^n$$

Now, using the following relationship:

$$\sum_{n=0}^{N} \alpha^n = \frac{1-\alpha^{(N+1)}}{1-\alpha}$$

We have:

$$\sum_{n=0}^{N} z^n = \sum_{n=0}^{N} \left(e^{i\theta}\right)^n = \frac{1-e^{i\theta(N+1)}}{1-e^{i\theta}} = \frac{e^{i\theta\left(\frac{N+1}{2}\right)}\left(e^{-i\theta\left(\frac{N+1}{2}\right)} - e^{i\theta\left(\frac{N+1}{2}\right)}\right)}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right)}$$

Now using Euler's identity, we can write the numerator and the denominator as follows:

$$e^{-i\theta\left(\frac{N+1}{2}\right)} - e^{i\theta\left(\frac{N+1}{2}\right)} = (-2i)\sin\left[\theta\left(\frac{N+1}{2}\right)\right]$$

$$e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} = (-2i)\sin\left(\frac{\theta}{2}\right)$$

By substituting these values and after simplification we get:

$$\sum_{n=0}^{N} z^n = \frac{\sin\left[\left(\frac{N+1}{2}\right)\theta\right]}{\sin\left(\frac{\theta}{2}\right)} \exp\left(i\frac{N\theta}{2}\right)$$

(b) With little algebraic manipulation, determine each of the following sums:

(i) $\displaystyle\sum_{n=0}^{N} \cos(n\,\theta)$

From euler's formula, we know that

$$\sum_{n=0}^{N} z^n = \sum_{n=0}^{N} e^{i\theta n} = \sum_{n=0}^{N} \cos(n\theta) + i\sum_{n=0}^{N} \sin(n\theta)$$

We only need to equate the real parts of first part:

$$\sum_{n=0}^{N} \cos(n\theta) = \begin{cases} N+1 & \text{if } \theta = 0 \\ \dfrac{\sin\left[\dfrac{(N+1)\theta}{2}\right]}{\sin\dfrac{\theta}{2}} \cos\left(\frac{N\theta}{2}\right) & \text{if } \theta \neq 0. \end{cases}$$

(ii) $\displaystyle\sum_{n=1}^{N} \sin(n\,\theta)$

Similarly, we only need to equate the imaginary parts of the first part:

$$\sum_{n=0}^{N} \sin(n\theta) = \begin{cases} 0 & \text{if } \theta = 0 \\ \dfrac{\sin\left[\dfrac{(N+1)\theta}{2}\right]}{\sin\dfrac{\theta}{2}} \sin\left(\frac{N\theta}{2}\right) & \text{if } \theta \neq 0. \end{cases}$$

The index on this identity starts from $(n = 1)$ because for $(n = 0)$ it evaluates to zero.

[OPTIONAL]

HW1.5 For each set defined below, provide a well-labeled diagram identifying all the points on the *complex plane* that belong to it. The symbol $\mathbb{C}$ refers to the set of complex numbers, $\mathbb{R}$ to the set of real numbers, and $\mathbb{Z}$ to the set of integers.

(a) $\{z \in \mathbb{C} \mid |z - i| = |z + i|\}$

$\{z \in \mathbb{C} | |z - i| = |z + i|\}$ is the set of all complex numbers that are equidistant from i and $-i$, which is equivalent to the real axis (Figure 3a).

(b) $\{z \in \mathbb{C} \mid \mathsf{Im}(z) > \mathsf{Re}(z)\}$

$\{z \in \mathbb{C} \mid \mathsf{Im}(z) > \mathsf{Re}(z)\}$ is the half of the complex plane above the line $\mathsf{Im}(z) = \mathsf{Re}(z)$ (Figure 3b).

(c) $\{z \in \mathbb{C} \mid 0 < \angle z < \pi/4\}$

$\{z \in \mathbb{C} \mid 0 < \angle z < \pi/4\}$ is a wedge of the complex plane (Figure 3c).

(d) $\{z \in \mathbb{C} \mid z + z^* = 0\}$

Note that $z + z^* = 2Re(z)$, so $\{z \in \mathbb{C} \mid z + z^* = 0\}$ is the set of points whose real part is 0. In other words, the imaginary axis (Figure 3d).

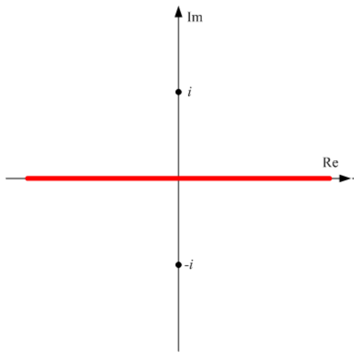
(e) $\{z \in \mathbb{C} \mid z = e^{i(2\pi/3)t}, t \in \mathbb{R}\}$

$\left\{z \in \mathbb{C} \mid z = e^{i(2\pi/3)t}, t \in \mathbb{R}\right\}$ is the unit circle. The $2\pi/3$ can be replaced by any non-zero angular frequency, and it would still be the unit circle (Figure 3e).

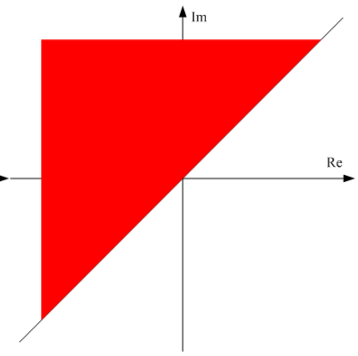
(f) $\{z \in \mathbb{C} \mid z = e^{i(2\pi/3)n}, n \in \mathbb{Z}\}$

$\left\{z \in \mathbb{C} \mid z = e^{i(2\pi/3)n}, n \in \mathbb{Z}\right\}$ is a set consisting of 3 distinct points (Figure 3f).

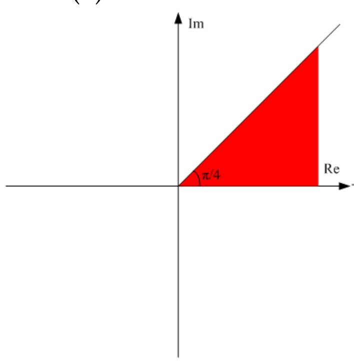
Note that all strict inequalities, i.e., $<$ as opposed to $\leq$, should be represented by dotted lines.



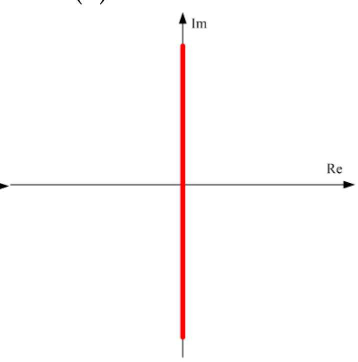
(a) Problem 1.5a.



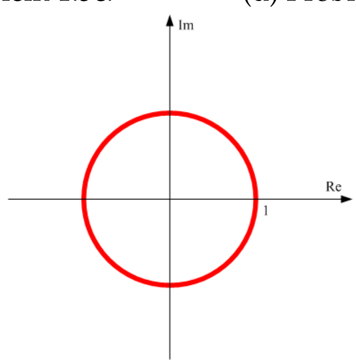
(b) Problem 1.5b.



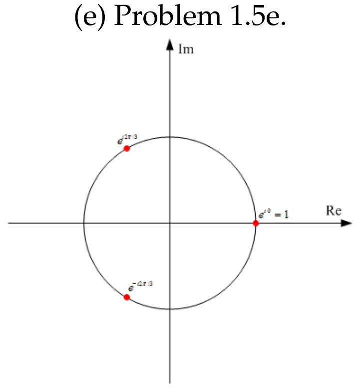
(c) Problem 1.5c.



(d) Problem 1.5d.



(e) Problem 1.5e.



(f) Problem 1.5f.

[OPTIONAL]

HW1.6 This problem explores some consequences of Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(a) $\cos\,\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$
$\cos\,\theta = \cos\,\theta + i\frac{1}{2}\sin\,\theta - i\frac{1}{2}\sin\,\theta = \frac{\cos\,\theta + i\,\sin\,\theta}{2} + \frac{\cos\,\theta + i\,\sin\,-\theta}{2} = \frac{e^{i\theta}+e^{-i\theta}}{2}$

(b) $\sin\,\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$
$\sin\,\theta = \sin\,\theta + \frac{1}{2i}\cos\,\theta - \frac{1}{2i}\cos\,\theta = \frac{\cos\,\theta + i\,\sin\,\theta}{2i} + \frac{-\cos\,\theta + i\,\sin\,\theta}{2i} = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

(c) Derive De Moivre's Theorem, where for all integers n,

$$(\cos\theta + i\,\sin\theta)^n = \cos(n\,\theta) + i\,\sin(n\,\theta).$$

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{i(n\theta)} = \cos\left(n\theta\right) + i\sin\left(n\theta\right)$$

(d) Using Euler's Formula with a strategically chosen value of θ, derive what American mathematician Benjamin Peirce referred to as the "mysterious formula" [1],

$$\pi = \frac{2}{i}\ln(i).$$

$$e^{i\left(\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

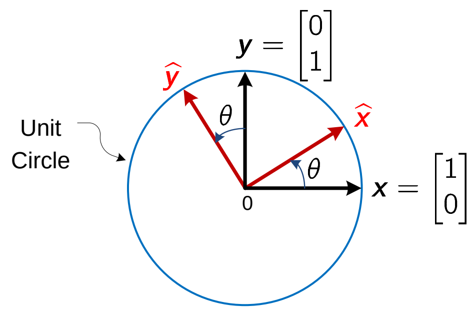
$$\ln\left(e^{i\left(\frac{\pi}{2}\right)}\right) = i\left(\frac{\pi}{2}\right) = \ln\left(i\right)$$

$$\pi = \frac{2}{i}\ln(i)$$

[1] For more on this, read the wonderful book by Paul J. Nahin, *An Imaginary Tale: The Story of* $\sqrt{-1}$, Princeton University Press, 1998, ISBN: 0-691-12798-0

[OPTIONAL]

HW1.7 (Multiplication by i) In this problem, we show that multiplying a complex number by i rotates its vectorial representation *counterclockwise* by the angle $\pi/2$. Consider the figure below, which shows the rotation of the canonical vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ through an angle θ.



(a) Determine reasonably simple forms for each of the rotated vectors $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$.

We can project $\hat{x}$ and $\hat{y}$ onto the real and imaginary axes, giving us

$$\hat{x} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } \hat{y} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(b) Determine reasonably simple forms for each of the products

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\boldsymbol{x}} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{y}}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

(c) Suppose $\boldsymbol{R}_\theta$ is the 2×2-matrix that rotates every vector in the two-dimensional plane by a counterclockwise angle θ. Let

$$\boldsymbol{R}_\theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then it must be that

$$\widehat{\boldsymbol{x}} = \boldsymbol{R}_\theta\, \boldsymbol{x} \quad \text{and} \quad \widehat{\boldsymbol{y}} = \boldsymbol{R}_\theta\, \boldsymbol{y}.$$

Determine the matrix $\boldsymbol{R}_\theta$, where each entry of the matrix is a reasonably simple expression in terms of θ.

$$\mathbf{R}_\theta x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \hat{x}$$

so $a = \cos\theta$ and $c = \sin\theta$. Similarly,

$$\mathbf{R}_\theta y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \hat{y}$$

so $b = -\sin\theta$ and $d = \cos\theta$.

(d) Let z have the Cartesian representation $z = a + ib$, and a corresponding vectorial representation

$$\boldsymbol{z} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then $\widetilde{z} = iz = -b + ia$, and its vectorial representation is $\widetilde{\boldsymbol{z}} = \begin{bmatrix} -b \\ a \end{bmatrix}$.

Show that $\widehat{\boldsymbol{z}} = \boldsymbol{R}_{\pi/2}\, \boldsymbol{z} = \widetilde{\boldsymbol{z}}$. This establishes the result that we set out to prove.

$$\mathbf{R}_{\pi/\mathbf{2}} z = \begin{bmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \hat{z}$$