Problem Set 6
Due: Fri, 9 Oct 2026, 23:59
Accepted Thru: Sun, 11 Oct 2026, 23:59

Your LAST Name _____ **Your FIRST Name** _____

Your Collaborators

- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This homework covers:

- Continuous-Time Convolution
- Continuous-Time IAGs, Systems, Stability

HW6.1 (Continuous-Time Convolution) The following input signal is applied to a real, continuous-time LTI system:

$$x : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \qquad x(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

For each LTI system described below, determine the output signal y in response to the input signal x; do this by providing a well-labeled plot of y as a function of t or finding an expression for y in terms of x.

(a) The impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \qquad h(t) = \begin{cases} 1 & \text{if } 1 \leq t \leq 2 \\ 0 & \text{elsewhere.} \end{cases}$$

We know that $y(t) = (x * h)(t)$. We are convolving two boxes of magnitude 1 and width 1. Since they are equally wide, $y(t)$ will be the triangular figure shown in Figure 1.

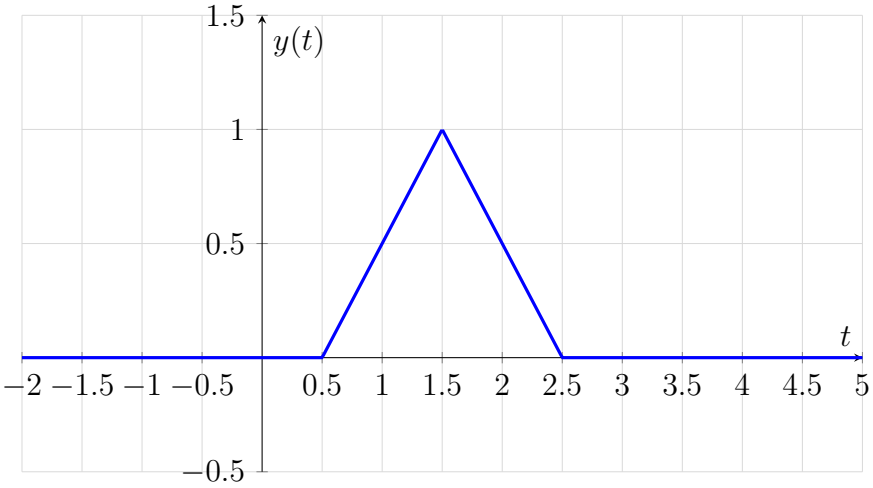


Figure 1: Plot for 4.4(a)

(b) The impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \qquad h(t) = \begin{cases} 1 & \text{if } 1 \leq t \leq 3 \\ 0 & \text{elsewhere.} \end{cases}$$

For this part, we are convolving of two boxes of magnitude 1, but differing widths. This will give us the saturated triangular figure shown below.

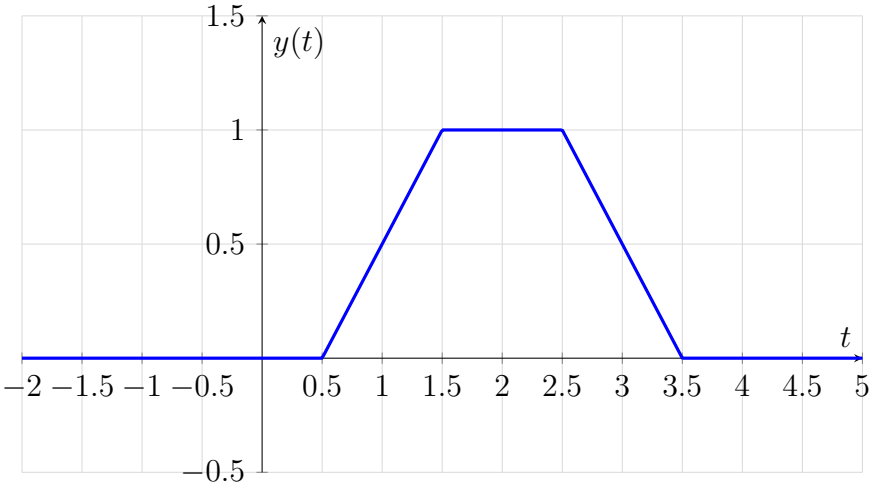


Figure 2: Plot for 4.4(b)

(c) The impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \qquad h(t) = \delta(t) - \delta\left(t - \frac{1}{2}\right),$$

where δ is the Dirac delta function.

Note that convolution with the Dirac delta just makes a copy of $x(t)$ centered on the active point for the Dirac delta scaled by the area of the delta: $(x * \alpha\delta(t - T))(t) = \alpha x(t - T)$. To get $y(t)$, we convolve $x(t)$ with $h(t)$:

$$\begin{aligned} y(t) &= (x * h)(t) \\ &= x(t) * (\delta(t) - \delta(t - \tfrac{1}{2})) \\ &= x(t) * \delta(t) - x(t) * \delta(t - \tfrac{1}{2}) \\ &= x(t) - x(t - \tfrac{1}{2}) \end{aligned}$$

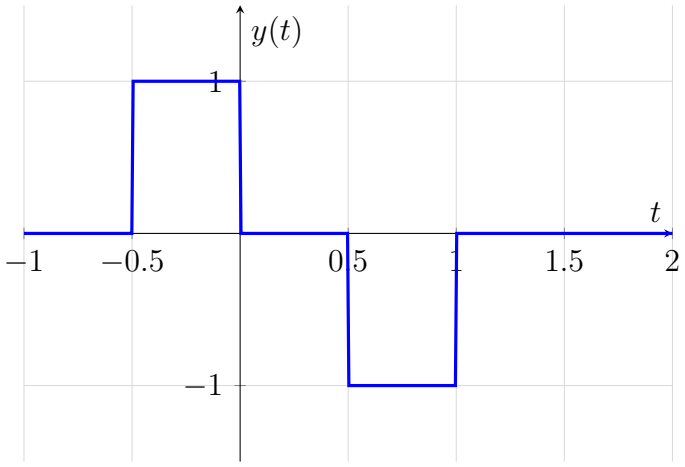


Figure 3: Plot for 4.4(c)

(d) The impulse response h of the LTI system is

$$h : \mathbb{R} \to \mathbb{R}$$
$$\forall t \in \mathbb{R}, \qquad h(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

where $1 < T$.

In this case, we simply have an infinite number of copies of our box, centered on the active point for each delta.

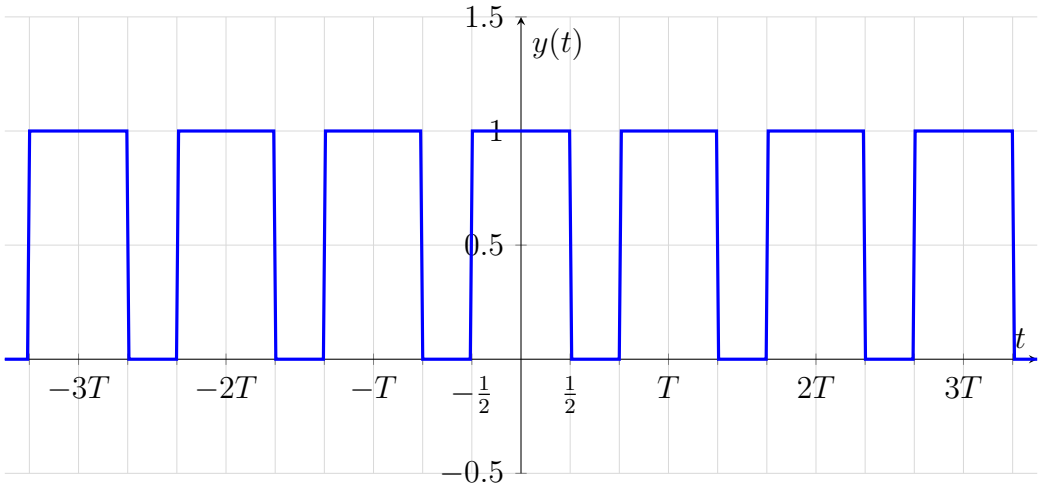
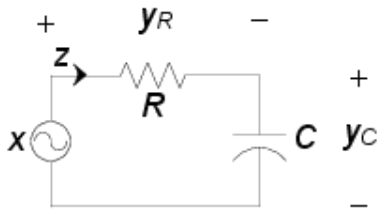


Figure 4: Plot for 4.4(d)

HW6.2 (Analysis of a First-Order Electronic Circuit)

In this problem, you will explore time- and frequency-domain descriptions of the input-output behavior of a first-order RC-circuit, shown below.



The signal x represents an input voltage. Each of y_R and y_C denotes the voltage drop across the resistor and capacitor, respectively, and represents a possible choice of an output signal.[1]

The signal denoted by z is the loop current that passes through the resistor and the capacitor.

The resistance R and the capacitance C are real constants having units of Ohms and Farads, respectively.

In the time domain, the input-output behavior of the system can be described by a linear, constant-coefficient differential equation. If the circuit is at initial rest—that is, if the charge on the capacitor is zero—then the circuit is an LTI system.

The differential equation governing the input x and the output y_C is:

$$RC\,\dot{y_C}(t) + y_C(t) = x(t)\,.$$

Here, we have made implicit use of the fact that

$$z(t) = \frac{x(t) - y_C(t)}{R}\ ,\ \text{for all } t$$

In lecture we found an expression for $H_C(\omega)$, $\forall \omega \in \mathbb{R}$, values of the frequency response H_C governing the relationship between the input x and the output y_C.

$$H_C(\omega) = \frac{1}{1 + RCi\omega}$$

It is a low pass filter noting that for ω near 0, $|H_C(\omega)| \approx 1$, while for larger values of $|\omega|$, $|H_C(\omega)|$ gets smaller.

[1] More generally, we can consider the vector $\begin{bmatrix} y_R \\ y_C \end{bmatrix} : \mathbb{R} \to \mathbb{R}^2$ as a vector output signal. Such considerations form the subject matter of more advanced courses. Here, we consider primarily single-input, single-output (SISO) models.

(a) Consider as the output the voltage drop y_R across the resistor.

(i) The total voltage drop y_R+y_C across the resistor and the capacitor equals the voltage difference x across the two nodes of the input source—that is, $x = y_R + y_C$. Show that the linear, constant-coefficient differential equation governing the input-output pair (x, y_R) of the circuit is

$$RC\,\dot{y_R}(t) + y_R(t) = RC\,\dot{x}(t)\,.$$

We can determine the differential equation as follows:

$$\begin{aligned} x &= y_R + y_C \\ \dot{x}(t) &= \dot{y_R}(t) + \dot{y_C}(t) \\ RC\dot{x}(t) &= RC\dot{y_R}(t) + RC\dot{y_C}(t). \end{aligned}$$

Noticing, from the differential equation in part (a), that $RC\dot{y_C}(t) = x(t) - y_C(t) = y_R(t)$,

$$RC\dot{x}(t) = RC\dot{y_R}(t) + y_R(t).$$

(ii) Determine an expression for the frequency response $H_R(\omega)$ governing the frequency-domain relationship between the input x and the output y_R.

With input $x(t) = e^{i\omega t}$, we have $y_R(t) = H_R(\omega)e^{i\omega t}$. Substituting these expressions into the differential equation from part (i), we get:

$$RCi\omega H_R(\omega)e^{i\omega t} + H_R(\omega)e^{i\omega t} = RCi\omega e^{i\omega t}$$

$$\implies H_R(\omega) = \frac{RCi\omega}{1 + RCi\omega}$$

(iii) By performing graphical analysis in the complex plane, provide well-labeled sketches, $\forall\omega \in \mathbb{R}$, of the magnitude response values $|H_R(\omega)|$ and phase response values $\angle H_R(\omega)$ of the circuit.

$$|H_R(\omega)| = \frac{|RCi\omega|}{|1 + RCi\omega|} = \frac{|(RCi\omega + 1) - 1|}{|1 + RCi\omega|}$$

$$\angle H_R(\omega) = \angle(RCi\omega) - \angle(1 + RCi\omega) = \angle((RCi\omega + 1) - 1) - \angle(1 + RCi\omega)$$

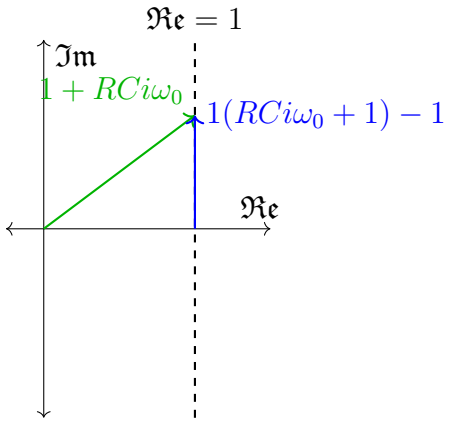
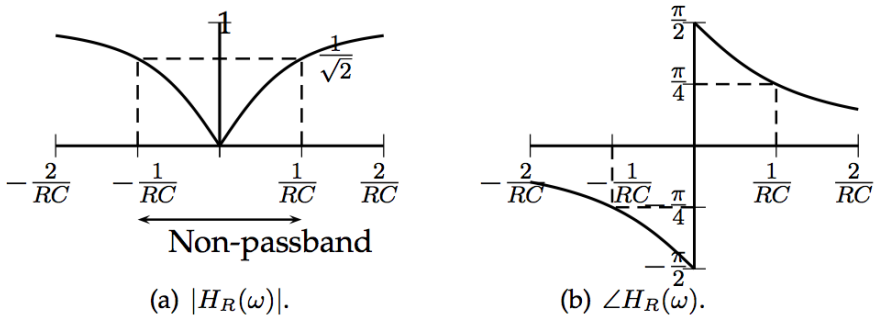


Figure 5: $|H_R(\omega)|$ depicted as a ratio of vector magnitudes at $\omega = \omega_0$ for 4.3(b)(iii)

At $\omega = 0$, $H_R(\omega) = 0$. As $|\omega|$ increases, the imaginary component of the denominator increases, and $H_R(\omega)$ approaches $\frac{RCi\omega}{RCi\omega} = 1$.
As $H_R(\omega) \to 1$ as $|\omega| \to \infty$, the phase approaches 0 as $\omega \to \pm\infty$. At $\omega = 0$, the phase is undefined, as $H_R(\omega) = 0$. For a small value of ω, $|\omega| = \epsilon$ such that $RC\epsilon \ll 1$, the real component of the denominator is much larger than the imaginary component. So, the phase of the denominator is approximately 0, and the phase of the numerator is $\frac{\pi}{2}$ for $\omega > 0$ and $-\frac{\pi}{2}$ for $\omega < 0$.



(a) $|H_R(\omega)|$. (b) $\angle H_R(\omega)$.

Figure 6: Plots for 4.3(b)(iii).

(iv) For this choice of input and output signals, is the filter low-pass, band-pass, high-pass, or none of these? Explain your reasoning succinctly, but clearly and convincingly.

It is a high pass filter noting that for ω near 0, $|H_R(\omega)| \approx 0$, while for larger values of $|\omega|$, $|H_R(\omega)|$ gets larger.

(v) Provide a well-labeled integrator-adder-gain diagram representation of the circuit, corresponding to the input-output pair (x, y_R). Your diagram must have the minimal number of storage elements (integrators) necessary to represent the filter.
$y_R(t) = -\frac{1}{RC}\int y_R(t)\,dt + x(t)$.

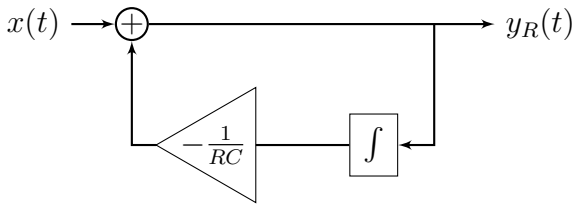


Figure 7: IAG Block Diagram for 4.3(b)(v)

(b) Using, once again, the fact that the total voltage drop $y_R + y_C$ across the resistor and the capacitor equals the voltage difference x across the two nodes of the input source, that is, $x = y_R + y_C$, and letting $x(t) = e^{i\omega t}$, show that

$$H_R(\omega) + H_C(\omega) = 1,$$

and use this as a "sanity check" for the expressions you obtained for $H_R(\omega)$ as well as $H_C(\omega)$ which was derived in lecture and given above.

If $x(t) = e^{i\omega t}$, then $y_R(t) = H_R(\omega)e^{i\omega t}$ and $y_C(t) = H_C(\omega)e^{i\omega t}$:

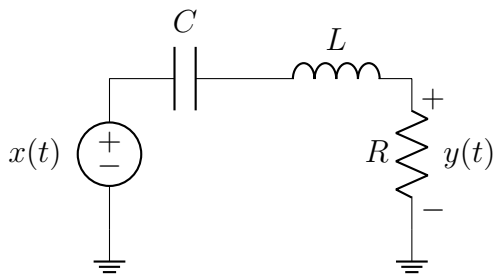
$$x = y_R + y_C \implies e^{i\omega t} = H_R(\omega)e^{i\omega t} + H_C(\omega)e^{i\omega t} \implies 1 = H_R(\omega) + H_C(\omega).$$

To check our expressions for $H_R(\omega)$ and $H_C(\omega)$, [2]

$$H_C(\omega) + H_R(\omega) = \frac{1}{1 + RCi\omega} + \frac{RCi\omega}{1 + RCi\omega} = 1.$$

[2] Later, you'll learn to apply the continuous-time Fourier transform to verify that $H_R(\omega) + H_C(\omega) = 1$.

HW6.3 RLC Filtering Suppose you have a system H represented by the following circuit, which applies a filter onto a constant signal $x(t)$ to get output $y(t)$. It is also given that R, L, C are positive constants.



This system is represented in the time domain by the following LCCDE:

$$\ddot{y}(t) + \frac{R}{L}\dot{y}(t) + \frac{1}{LC}y(t) = 0.$$

(a) The state-space representation of the system can be represented as

$$\begin{bmatrix} \dot{q_1}(t) \\ \dot{q_2}(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

where $q_1(t) = y(t)$ and $q_2(t) = \dot{y}(t)$. Find the matrix $\mathbf{A}$.

We can formulate two equations

$$\dot{q_1}(t) = q_2(t)$$

$$\dot{q_2}(t) = -\frac{1}{LC}q_1(t) - \frac{R}{L}q_2(t)$$

Putting this into matrix-vector form, we get

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}$$

(b) What values of R should we use if we don't want the system to have any oscillatory behavior (i.e we want it to be critically damped or overdamped)? Write your answer in terms of L and C.

The eigenvalues of this matrix are

$$\lambda_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

$$\lambda_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

We need the eigenvalues to be entirely real, so the expression inside the square root must not be negative.

$$\frac{R^2}{L^2} - \frac{4}{LC} \geq 0 \implies R \geq 2\sqrt{\frac{L}{C}}$$

(c) What values of R, L and C should we use if we want our system to be internally stable?

In order for the system to be internally stable, we need $\text{Re}\{\lambda_1\} < 0$ and $\text{Re}\{\lambda_2\} < 0$. Since the square root of a number is always either a non-negative real number or a purely imaginary number, λ_2 will have a real part smaller than 0 for all values of R, L and C. For λ_1, we don't want

$$-\frac{R}{L} + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \geq 0 \implies \frac{4}{LC} \leq 0 \implies \frac{1}{LC} = 0$$

The last step is because we are given that LC is strictly positive.

Therefore, we require that $L \neq \infty$, $C \neq \infty$ for the system to be internally stable. R can take on any value.