Problem Set 7
Due: Fri, 16 Oct 2026, 23:59
Accepted Thru: Sun, 18 Oct 2026, 23:59

Your LAST Name _____ **Your FIRST Name** _____

Your Collaborators

- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This homework covers:

- State Space
- Feedback
- DTFS

NOTE: This is a repeat problem from HW5. The solutions for this problem have been released but please try the problem earnestly. Parts (a) and (b) are optional and can be reused from the previous homework. Parts (c), (d), and (e) are mandatory.

HW7.1 (A Jolly Famous Sequence of Numbers)

The input x and the output y of a causal, discrete-time LTI system H are governed by the second-order linear constant-coefficient difference equation

$$y[n] = y[n-1] + y[n-2] + x[n],$$

where the system is assumed to be initially at rest—that is, $y[n] = 0$ for all $n < 0$.

(a) Determine $h[n]$, the impulse response values of the system H. What famous sequence of numbers does the impulse response h trace? Is the system BIBO stable? Provide a succinct, yet clear and convincing explanation.

Plug in $x[n] = \delta[n]$.

$$\begin{aligned}
&\mathsf{H}\text{ causal} \implies h[n] = 0, \quad \forall n < 0 \\
&h[0] = h[-1] + h[-2] + \delta[0] = 1 \\
&h[1] = h[0] + h[-1] + \underbrace{\delta[1]}_{0} = 1 \\
&h[2] = h[1] + h[0] = 2 \\
&h[3] = h[2] + h[1] = 3 \\
&h[4] = h[3] + h[2] = 5 \\
&\vdots
\end{aligned}$$

The impulse response is the Fibonacci sequence.

This system is not BIBO stable, as the impulse response keeps on growing with each sample and is therefore not finitely summable.

(b) Draw a delay-adder-gain (DAG) block diagram implementation of the system.

Because of the $y[n-2]$ term, at least two delay blocks are needed to implement the system. There are a couple DAGs that can be drawn for this problem, with two of them shown below.

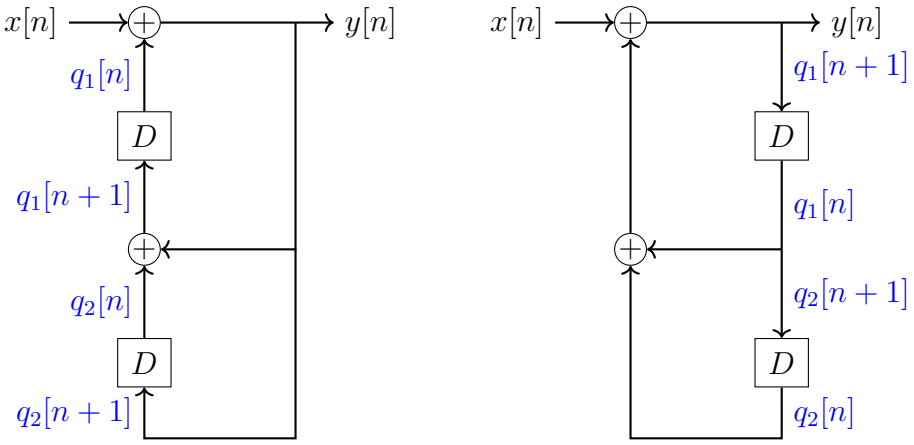


Figure 1: Two possible delay-adder-gain block diagrams for 4.1(b)

(c) On the DAG block diagram implementation of the previous part, label the appropriate state variables $q_1[n]$ and $q_2[n]$. Then proceed to write the corresponding state-space equations

$$\boldsymbol{q}[n+1] = \mathbf{A}\,\boldsymbol{q}[n] + \mathbf{B}\,x[n] \qquad \text{State-Evolution Equation}$$

and

$$y[n] = \mathbf{C}\,\boldsymbol{q}[n] + \mathbf{D}\,x[n], \qquad \text{Output Equation}$$

where $\boldsymbol{q}[n] = [q_1[n] \;\; q_2[n]]^\mathsf{T}$ is the state vector at time n. You must, of course, determine the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ completely.

We denote where our state variables are on the diagrams in Figure 1. For the diagram on the left,

$$q_1[n] \triangleq y[n-1] + y[n-2]$$
$$q_2[n] \triangleq y[n-1]$$
$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} y[n] + q_2[n] \\ y[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{B}} x[n]$$
$$y[n] = q_1[n] + x[n] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \vec{q}[n] + Dx[n], \quad D \triangleq 1$$

For the diagram on the right,

$$q_i[n] \triangleq y[n-i], i = 1, 2$$
$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} y[n] \\ y[n-1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{B}} x[n]$$
$$y[n] = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\mathbf{C}} \vec{q}[n] + x[n]$$

Note that if we swap our definitions for $q_1[n]$ and $q_2[n]$ for either DAG block diagram, we will get $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, which is a valid setup but will not match the format for the modes in part (d).

(d) Determine the modes $(\lambda_1, \boldsymbol{v}_1)$ and $(\lambda_2, \boldsymbol{v}_2)$ of the system. For consistency, show that the modes are of the form $\left(\lambda_1, \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -\lambda_2 \end{bmatrix}\right)$ and $\left(\lambda_2, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -\lambda_1 \end{bmatrix}\right)$, where λ_1 and λ_2 are the eigenvalues that you must determine. Also, explain why the eigenvectors *must* be mutually orthogonal.

Since an eigenvalue λ and eigenvector $\vec{v} \neq \vec{0}$ of $\mathbf{A}$ satisfy $\mathbf{A}\vec{v} = \lambda\vec{v}$, $\lambda\mathbf{I} - \mathbf{A}$ has a nontrivial nullspace and $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

$$\det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} : (\lambda_1\mathbf{I} - \mathbf{A})\,\vec{v}_1 = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \vec{v}_1 = \vec{0}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\lambda_2 \end{bmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} : (\lambda_2\mathbf{I} - \mathbf{A})\,\vec{v}_2 = \begin{bmatrix} \frac{-1-\sqrt{5}}{2} & -1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \vec{v}_2 = \vec{0}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\lambda_1 \end{bmatrix}$$

The eigenvectors *must* be orthogonal because $\mathbf{A}$ is a symmetric matrix. By the Spectral Theorem, we can decompose a symmetric matrix: $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. $\mathbf{V}$ is an orthonormal matrix containing the eigenvectors of $\mathbf{A}$, so the eigenvectors of $\mathbf{A}$ are orthogonal.

(e) Is the system *internally stable*? Provide a succinct, yet clear and convincing explanation.

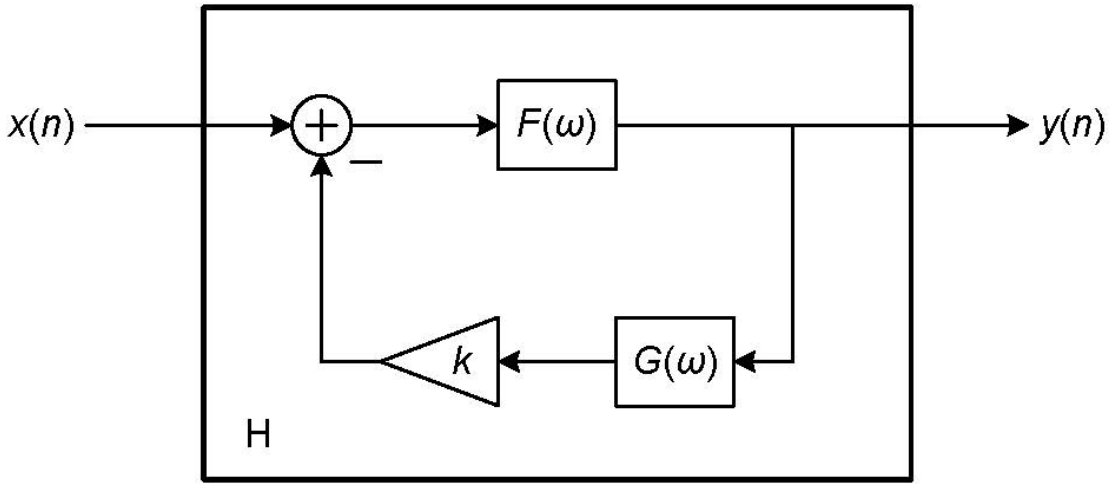
The system is not internally stable because the mode associated with λ_1 "blows up" as $n \to \infty$. Since $\mathbf{A}$ is diagonalizable, we can look at the state-evolution

equation under a change of basis.

$$
\begin{aligned}
\mathbf{A}\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} &= \underbrace{\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}}_{\triangleq \mathbf{V}} \underbrace{\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}}_{\triangleq \mathbf{D}} \\
\vec{q}[n+1] &= \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\vec{q}[n] \\
\underbrace{\mathbf{V}^{-1}\vec{q}[n+1]}_{\vec{z}[n+1]} &= \mathbf{D}\underbrace{\mathbf{V}^{-1}\vec{q}[n]}_{\triangleq \vec{z}[n]} \\
\vec{z}[n+1] &= \mathbf{D}\vec{z}[n] \\
\begin{bmatrix}z_1[n+1]\\ z_2[n+1]\end{bmatrix} &= \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}z_1[n]\\ z_2[n]\end{bmatrix} \\
&= \begin{bmatrix}\lambda_1 z_1[n]\\ \lambda_2 z_2[n]\end{bmatrix} \\
z_i[n+1] &= \lambda_i z_i[n], \quad i = 1, 2 \\
z_i[n] &= \lambda_i^{n-1} z_i[1]
\end{aligned}
$$

Since $|\lambda_1| = \frac{1+\sqrt{5}}{2} \approx 1.618 > 1$ and $z_1[1] \neq 0$, $\lim_{n\to\infty} |z_1[n]| = \infty$.

HW7.2 (Feedback System and BIBO Stability) Consider the discrete-time feedback system with frequency response $H(\omega)$, as shown in the figure. The LTI plant has frequency response $F(\omega)$. The LTI controller is the cascade of an LTI system with frequency response $G(\omega)$ and a tunable multiplier with multiplication factor k. Assume the overall system H is causal.



The plant has the impulse response values

$$f(n) = \begin{cases} \alpha^n & \text{if } n = 1, 3, 5, 7, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < 1$. Furthermore, $G(\omega) = e^{-i\omega}$

(a) Determine the frequency response $F(\omega)$ of the plant.

$$\begin{aligned} F(\omega) &= \sum_{n\in\mathbb{Z}} f(n)e^{-in\omega} \\ &= \sum_{n\geq 0,\ n \text{ odd}} \alpha^n e^{-in\omega} \\ &= \alpha e^{-i\omega} \sum_{l=0}^{\infty} (\alpha^2 e^{-i2\omega})^l \\ &= \frac{\alpha e^{-i\omega}}{1 - \alpha^2 e^{-i2\omega}} \,. \end{aligned}$$

(b) Determine the impulse response values $g(n)$ corresponding to $G(\omega)$.

$G(\omega)$ is recognized as the frequency response of the unit delay. Its impulse response is therefore

$$g(n) = \delta(n-1) \; .$$

This can be verified by checking that

$$e^{-i\omega} = \sum_{n\in\mathbb{Z}} \delta(n-1)e^{-in\omega} \; .$$

(c) What is the forward gain of the feedback system?

The forward gain is

$$F(\omega) = \frac{\alpha e^{-i\omega}}{1-\alpha^2 e^{-i2\omega}} \; .$$

(d) What is the loop gain of the feedback system?

The loop gain is

$$-kG(\omega)F(\omega) = -\frac{k\alpha e^{-i2\omega}}{1-\alpha^2 e^{-i2\omega}} \; .$$

(e) Determine the closed-loop frequency response $H(\omega)$. *You may derive it or apply Black's Formula.*

The closed loop frequency response is

$$\begin{aligned} H(\omega) &= \frac{\text{Forward Gain}}{1 - \text{ Loop Gain}} \\ &= \frac{F(\omega)}{1 + kG(\omega)F(\omega)} \\ &= \frac{\frac{\alpha e^{-i\omega}}{1-\alpha^2 e^{-i2\omega}}}{1 + \frac{k\alpha e^{-i2\omega}}{1-\alpha^2 e^{-i2\omega}}} \\ &= \frac{\alpha e^{-i\omega}}{1 - (\alpha^2 - k\alpha)e^{-i2\omega}} \,. \end{aligned}$$

(f) For what values of k is the closed loop system BIBO stable?

Hint: Your answer must be in terms of α. Recall that a discrete-time LTI system having impulse response h is BIBO stable if, and only if, $\sum_{n\in\mathbb{Z}} |h(n)| < \infty$.

From the formula for $H(\omega)$, we can derive its LCCDE:

$$\begin{aligned} H(\omega) - (\alpha^2 - k\alpha)H(\omega)e^{-i2\omega} &= \alpha e^{-i\omega} \\ H(\omega)e^{i\omega n} - (\alpha^2 - k\alpha)H(\omega)e^{-i2\omega}e^{i\omega n} &= \alpha e^{-i\omega}e^{i\omega n} \\ y(n) - (\alpha^2 - k\alpha)y(n-2) &= \alpha x(n-1) \\ h(n) - (\alpha^2 - k\alpha)h(n-2) &= \alpha\delta(n-1) \end{aligned}$$

Plugging in for a few values, we can see that:

$$\begin{aligned} h(0) &= 0 \\ h(1) &= \alpha \\ h(2) &= 0 \\ h(3) &= \alpha(\alpha^2 - k\alpha) \\ h(4) &= 0 \\ h(5) &= \alpha(\alpha^2 - k\alpha)^2 \end{aligned}$$

Unrolling the recursion, we can state that

$$h(n) = \begin{cases} 0 & \text{unless } n \geq 0 \text{ and } n \text{ is odd} \\ \alpha(\alpha^2 - k\alpha)^{(n-1)/2} & \text{if } n \geq 0 \text{ and } n \text{ is odd} \end{cases} .$$

Thus

$$\sum_{n\in\mathbb{Z}} |h(n)| = |\alpha| \sum_{l=0}^{\infty} |\alpha^2 - k\alpha|^l \ .$$

This means that

$$\sum_{n\in\mathbb{Z}} |h(n)| < \infty \Longleftrightarrow |\alpha^2 - k\alpha| < 1 \ .$$

Thus the closed loop system is BIBO stable if and only if

$$\alpha^2 - k\alpha < 1 \text{ and } \alpha^2 - k\alpha > -1 \ ,$$

i.e., if and only if

$$\alpha - \frac{1}{\alpha} < k < \alpha + \frac{1}{\alpha} \ .$$

HW7.3 (Inner Product) Let $\boldsymbol{x} \in \mathbb{C}^p$ and $\boldsymbol{y} \in \mathbb{C}^p$ be two p-dimensional vectors of complex numbers, thought of as column vectors, where p is a positive integer. The *inner product* of $\boldsymbol{x}$ with $\boldsymbol{y}$, denoted $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, is the complex number defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T \boldsymbol{y}^*.$$

Here $\boldsymbol{x}^T$ denotes the transpose of $\boldsymbol{x}$, and $\boldsymbol{y}^*$ denotes the complex conjugate of $\boldsymbol{y}$. For example, if

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{p-1} \end{bmatrix},$$

then

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{k=0}^{p-1} x_k \, y_k^*.$$

In the following, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \ldots, \boldsymbol{x_k}, \boldsymbol{y_1}, \boldsymbol{y_2}$ are all vectors in $\mathbb{C}^p$.

(a) Show that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is a real number. Further, show that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, with equality if, and only if, $\boldsymbol{x}$ is the zero vector.

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \sum_{i=1}^{n} x(i) x(i)^* = \sum_{i=1}^{n} |x(i)|^2 .$$

This is a sum of nonnegative real numbers, hence it is real and nonnegative. Further, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ equals zero if and only if each $|x(i)|^2$ equals zero, i.e. if and only if each $x(i)$ equals zero, i.e. if and only if $\boldsymbol{x}$ is the zero vector.

(b) Show that $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle^*$.

$$\langle \boldsymbol{y}, \boldsymbol{x} \rangle = \sum_{i=1}^{n} y(i)x(i)^* = \sum_{i=1}^{n} (x(i)y(i)^*)^* = (\sum_{i=1}^{n} x(i)y(i)^*)^* = (\langle \boldsymbol{x}, \boldsymbol{y} \rangle)^* \ .$$

(c) Show that $\langle \boldsymbol{x_1} + \boldsymbol{x_2}, \boldsymbol{y} \rangle = \langle \boldsymbol{x_1}, \boldsymbol{y} \rangle + \langle \boldsymbol{x_2}, \boldsymbol{y} \rangle$.

$$\begin{aligned} \langle \boldsymbol{x_1} + \boldsymbol{x_2}, \boldsymbol{y} \rangle &= \sum_{i=1}^{n} (x_1(i) + x_2(i))y(i)^* \\ &= \sum_{i=1}^{n} x_1(i)y(i)^* + \sum_{i=1}^{n} x_2(i)y(i)^* \\ &= \langle \boldsymbol{x_1}, \boldsymbol{y} \rangle + \langle \boldsymbol{x_2}, \boldsymbol{y} \rangle \end{aligned}$$

True or False? $\langle \boldsymbol{x}, \boldsymbol{y_1} + \boldsymbol{y_2} \rangle = \langle \boldsymbol{x}, \boldsymbol{y_1} \rangle + \langle \boldsymbol{x}, \boldsymbol{y_2} \rangle$. No explanation is necessary.

True.

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{y_1} + \boldsymbol{y_2} \rangle &= \sum_{i=1}^{n} x(i)(y_1(i) + y_2(i))^* \\ &= \sum_{i=1}^{n} x(i)y_1(i)^* + \sum_{i=1}^{n} x(i)y_2(i)^* \\ &= \langle \boldsymbol{x}, \boldsymbol{y_1} \rangle + \langle \boldsymbol{x}, \boldsymbol{y_2} \rangle \ . \end{aligned}$$

(d) We know that $\langle \alpha\, \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, where α is a complex scalar. Show that $\langle \boldsymbol{x}, \alpha\, \boldsymbol{y} \rangle = \alpha^* \langle \boldsymbol{x}, \boldsymbol{y} \rangle$.

$$\begin{aligned} \langle \boldsymbol{x}, \alpha \boldsymbol{y} \rangle &= \sum_{i=1}^{n} x(i)(\alpha y(i))^* \\ &= \sum_{i=1}^{n} x(i)\alpha^* y(i)^* = \alpha^* \sum_{i=1}^{n} x(i)y(i)^* \\ &= \alpha^* \langle \boldsymbol{x}, \boldsymbol{y} \rangle \ . \end{aligned}$$

(e) If $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, we say that $\boldsymbol{x}$ is *orthogonal* to $\boldsymbol{y}$, and we denote this by $\boldsymbol{x} \perp \boldsymbol{y}$. According to part ((b)), this implies that $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = 0$. Show that if $\boldsymbol{x}$ is orthogonal to $\boldsymbol{y}$, then

$$\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{y}, \boldsymbol{y} \rangle.$$

Explain how this is an articulation of Pythagoras's theorem.

Since $\boldsymbol{x}$ is orthogonal to $\boldsymbol{y}$, we have $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, by definition. By part (b) of the problem, this also means that $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = 0$. Hence we can write, using parts (c) and (d) of the problem

$$\begin{aligned} \langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \rangle &= \langle \boldsymbol{x}, \boldsymbol{x} + \boldsymbol{y} \rangle + \langle \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{y}, \boldsymbol{x} \rangle + \langle \boldsymbol{y}, \boldsymbol{y} \rangle \\ &\overset{(a)}{=} \langle \boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{y}, \boldsymbol{y} \rangle \ . \end{aligned}$$

Here, in step (a), we used the orthogonality of $\boldsymbol{x}$ and $\boldsymbol{y}$ (this step is not correct in general).

The Pythagorean theorem says that $a^2 + b^2 = c^2$ where a and b are orthogonal sides of a right triangle. In the inner product space, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \|\boldsymbol{x}\|^2$, which provides the connection between the inner product space and the Pythagorean theorem when applied to the equation we just derived.

(f) Suppose $\boldsymbol{x} = \alpha_1 \boldsymbol{x_1} + \ldots + \alpha_k \boldsymbol{x_k}$, for some complex scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$. Show that if

$$\langle \boldsymbol{x}, \boldsymbol{x_i} \rangle = 0 \quad \text{for all } 1 \leq i \leq k,$$

then $\boldsymbol{x}$ must be the zero vector.

We write

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{x} \rangle &= \langle \boldsymbol{x}, a_1 \boldsymbol{x_1} + a_2 \boldsymbol{x_2} + \ldots + a_k \boldsymbol{x_k} \rangle \\ &= \langle \boldsymbol{x}, a_1 \boldsymbol{x_1} \rangle + \langle \boldsymbol{x}, a_2 \boldsymbol{x_2} \rangle + \ldots + \langle \boldsymbol{x}, a_k \boldsymbol{x_k} \rangle \\ &= a_1^* \langle \boldsymbol{x}, \boldsymbol{x_1} \rangle + a_2^* \langle \boldsymbol{x}, \boldsymbol{x_2} \rangle + \ldots + a_k^* \langle \boldsymbol{x}, \boldsymbol{x_k} \rangle \\ &= 0 \ , \end{aligned}$$

where, in the last step we use the given fact that $\boldsymbol{x}$ is orthogonal to each of the $\boldsymbol{x_i}$, $1 \leq i \leq k$. Now, using part (a) of the problem, we know that $\boldsymbol{x}$ must be the zero vector.

HW7.4 (DTFS Expansion) Determine the complex-exponential discrete-time Fourier series (DTFS) expansion for each signal $x : \mathbb{Z} \to \mathbb{R}$ described below, or explain why no such expansion exists. For each case where a DTFS expansion exists, be sure to identify the period p and the fundamental frequency ω_0.

For parts (f) and (g), assume x denotes *exactly one period* of a periodic signal $\widetilde{x}$. Your answer would then be the DTFS expansion of the periodic signal $\widetilde{x}$ for certain values of n, and zero otherwise.

(a) $x(n) = \sin\left(\frac{2\pi}{5}n\right) + \cos\left(\frac{4\pi}{5}n\right), \quad \forall n.$

$$\begin{aligned} x(n) &= \sin(\frac{2\pi}{5}n) + \cos(\frac{4\pi}{5}n) \\ &= \frac{1}{2i}(e^{i\frac{2\pi}{5}n} - e^{-i\frac{2\pi}{5}n}) + \frac{1}{2}(e^{i\frac{4\pi}{5}n} + e^{-i\frac{4\pi}{5}n}) \end{aligned}$$

Therefore, $p = 5$, $\omega_0 = \frac{2\pi}{5}$.

(b) $x(n) = \cos\left(\frac{3\pi}{5}n\right) + \frac{1}{3}\cos\left(\frac{4\pi}{5}n\right), \quad \forall n.$

$$\begin{aligned} x(n) &= \cos(\frac{3\pi}{5}n) + \frac{1}{3}\cos(\frac{4\pi}{5}n) \\ &= \frac{1}{2}(e^{i\frac{3\pi}{5}n} + e^{-i\frac{3\pi}{5}n}) + \frac{1}{6}(e^{i\frac{4\pi}{5}n} + e^{-i\frac{4\pi}{5}n}) \end{aligned}$$

Therefore, $p = 10$, $\omega_0 = \frac{\pi}{5}$.

(c) $x(n) = \cos\left(\frac{\sqrt{2}\,\pi}{5}n\right), \quad \forall n.$

$$\begin{aligned} x(n) &= \cos(\frac{\sqrt{2}\pi}{5}n) \\ \frac{\sqrt{2}\pi}{5}p &= 2k\pi \iff p = 5\sqrt{2}k \end{aligned}$$

Since $\sqrt{2}$ is irrational, p cannot be an integer, and hence x is not periodic.

(d) $x(n) = \cos\left(\frac{2\pi}{3}n\right) + (-1)^n, \quad \forall n.$

$$\begin{aligned} x(n) &= \cos(\frac{2\pi}{3}n) + (-1)^n = \cos(\frac{2\pi}{3}n) + e^{i\pi n} \\ &= \frac{1}{2}(e^{i\frac{2\pi}{3}n} + e^{-i\frac{2\pi}{3}n}) + e^{i\pi n} \end{aligned}$$

Therefore, $p = 6$, $\omega_0 = \frac{\pi}{3}$.