Problem Set 8
Due: Fri, 23 Oct 2026, 23:59
Accepted Thru: Sun, 25 Oct 2026, 23:59

Your LAST Name _____ **Your FIRST Name** _____

Your Collaborators

- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____
- Name: _____ SID: _____

Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This homework covers:

- DTFS
- DFT

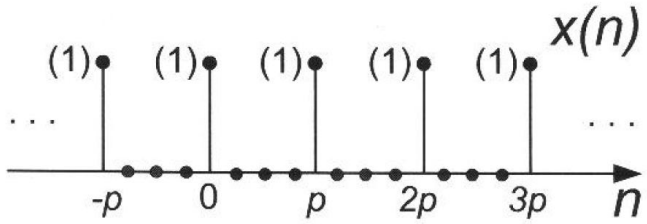
HW8.1 (DTFS Expansion) Determine the complex-exponential discrete-time Fourier series (DTFS) expansion for each signal $x : \mathbb{Z} \to \mathbb{R}$ described below, or explain why no such expansion exists. For each case where a DTFS expansion exists, be sure to identify the period p and the fundamental frequency ω_0.

For parts (b) and (c), assume x denotes *exactly one period* of a periodic signal $\widetilde{x}$. Your answer would then be the DTFS expansion of the periodic signal $\widetilde{x}$ for certain values of n, and zero otherwise.

(a) $x[n] = \sum_{l=-\infty}^{+\infty} \delta[n - lp]$, where p is a positive integer.

$$x(n) = \sum_{l=-\infty}^{\infty} \delta(n - lp)$$

A graphical representation of $x(n)$ is shown below: (Note p is the period and $\omega_0 = \frac{2\pi}{p}$ is the fundamental frequency)



The Discrete-time Fourier Series (DTFS) equations are :

$x[n] = \sum_{k=\langle p \rangle} X_k e^{ikw_0 n}$ Synthesis Equation

$X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) e^{-ikw_0 n}$ Analysis Equation

Note: $\langle p \rangle$ denotes an appropriately-chosen interval of p adjacent samples, e.g., $\{0, 1, \ldots, p-1\}$ or $\{1, 2, \ldots, p\}$,etc.

Let $\langle p \rangle = \{0, 1, \ldots, p-1\}$. Note that $x(n) = \delta(n)$ in the interval $\langle p \rangle$ that we have chosen. Therefore, only one term in the right-hand side of the analysis

equation is nonzero (the term corresponding to $n = 0$):

$$\begin{aligned} X_k &= \frac{1}{p}\sum_{n=0}^{p-1} x(n)e^{-ikw_0 n} \\ &= \frac{1}{p}\left(\underbrace{x(0)}_{=1} + \underbrace{x(1)}_{=0} e^{-ikw_0} + \ldots + \underbrace{x(p-1)}_{=0} e^{-ik(p-1)w_0}\right) \\ \Rightarrow \quad & X_k = \frac{1}{p}, \forall k \in \mathbb{Z} \Rightarrow x(n) = \frac{1}{p}\sum_{k=0}^{p-1} e^{ikw_0 n} \end{aligned}$$

This shows that all frequency components $\{0, w_0, \alpha w_0, \ldots, (p-1)w_0\}$ are equally present in the impulse train signal $x(n)$.

(b) $x[n] = \delta[n+2] + 2\delta[n+1] + 3\,\delta[n] + 2\,\delta[n-1] + \delta[n-2]$.

Since the signal x is of finite length 5, it can be extended to a periodic signal $\hat{x}$ with period $p = 5$.

$$\hat{x}(n) = \sum_{l=-\infty}^{l=\infty} x(n-5l)$$

$\hat{x}(n) = \sum_{l=-\infty}^{l=\infty} \delta(n+2-5l)+2\delta(n+1-5l)+3\delta(n-5l)+2\delta(n-1-5l)+\delta(n-2-5l)$.
Using the DTFS analysis equation, we have

$$\hat{X}_k = \frac{1}{p}\sum_{n=\langle p\rangle} \hat{x}(n)e^{-ik\omega_0 n},$$

where $\omega_0 = \frac{2\pi}{5}$ is the fundamental frequency. So,

$$\begin{aligned}
\hat{X}_0 &= \frac{1}{5}(\hat{x}(-2)e^0 + \hat{x}(-1)e^0 + \hat{x}(0)e^0 + \hat{x}(1)e^0 + \hat{x}(2)e^0) = 1.8 \\
\hat{X}_1 &= \frac{1}{5}(e^{2i\omega_0} + 2e^{i\omega_0} + 3 + 2e^{-i\omega_0} + e^{-2i\omega_0}) \\
&= \frac{1}{5}(e^{\frac{4\pi i}{5}} + 2e^{\frac{2\pi i}{5}} + 3e^0 + 2e^{\frac{-2\pi i}{5}} + e^{\frac{-4\pi i}{5}}) \\
&= \frac{1}{5}(2\cos(\frac{4\pi}{5}) + 4\cos(\frac{2\pi}{5}) + 3) \\
\hat{X}_2 &= \frac{1}{5}(e^{\frac{8\pi i}{5}} + 2e^{\frac{4\pi i}{5}} + 3 + 2e^{\frac{-4\pi i}{5}} + e^{\frac{-8\pi i}{5}}) = \frac{1}{5}(2\cos(\frac{8\pi}{5}) + 4\cos(\frac{4\pi}{5}) + 3) = 0.07639 \\
\hat{X}_3 &= \frac{1}{5}(e^{\frac{12\pi i}{5}} + 2e^{\frac{6\pi i}{5}} + 3 + 2e^{\frac{-6\pi i}{5}} + e^{\frac{-12\pi i}{5}}) = \frac{1}{5}(2\cos(\frac{12\pi}{5}) + 4\cos(\frac{6\pi}{5}) + 3) \\
&= \frac{1}{5}(2\cos(\frac{8\pi}{5}) + 4\cos(\frac{4\pi}{5}) + 3) = \hat{X}_2 = 0.07639 \\
\hat{X}_4 &= \frac{1}{5}(e^{\frac{16\pi i}{5}} + 2e^{\frac{8\pi i}{5}} + 3 + 2e^{-\frac{8\pi i}{5}} + e^{\frac{-16\pi i}{5}}) = \frac{1}{5}(2\cos(\frac{16\pi}{5}) + 4\cos(\frac{8\pi}{5}) + 3) \\
&= \frac{1}{5}(2\cos(\frac{4\pi}{5}) + 4\cos(\frac{2\pi}{5}) + 3) = \hat{X}_1 = 0.52361.
\end{aligned}$$

Therefore, $x(n) = \begin{cases} \hat{X}_0 + \hat{X}_1 e^{\frac{2i\pi n}{5}} + \hat{X}_2 e^{\frac{4i\pi n}{5}} + \hat{X}_3 e^{\frac{6i\pi n}{5}} + \hat{X}_4 e^{\frac{8i\pi n}{5}} & -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$

Notice that all the DTFS coefficients can be expressed as cosines and are therefore purely real valued, and they are also symmetric (even). This is a consequence of conjugate symmetry - because x is real valued and even, the DTFS coefficients X are also real valued and even.

(c) $x[n] = \delta[n+2] + \delta[n+1] - \delta[n-1] - \delta[n-2]$.

Since the signal x is of finite length 5, it can be extended to a periodic signal $\hat{x}$ with period $p = 5$, $\omega_0 = \frac{2\pi}{5}$.

$$\hat{x}(n) = \sum_{l=-\infty}^{l=\infty} \delta(n+2-5l) + \delta(n+1-5l) - \delta(n-1-5l) - \delta(n-2-5l).$$

Using the DTFS analysis equation, we get

$$\begin{aligned}
\hat{X}_0 &= \frac{1}{5}(e^0 + e^0 + 0 - e^0 - e^0) = 0 \\
\hat{X}_1 &= \frac{1}{5}(e^{2i\omega_0} + e^{i\omega_0} - e^{-i\omega_0} - e^{-2i\omega_0}) = \frac{2i}{5}(\sin(\frac{4\pi}{5}) + \sin(\frac{2\pi}{5})) = 0.615537i \\
\hat{X}_2 &= \frac{1}{5}(e^{4i\omega_0} + e^{2i\omega_0} - e^{-2i\omega_0} - e^{-4i\omega_0}) = \frac{2i}{5}(\sin(\frac{8\pi}{5}) + \sin(\frac{4\pi}{5})) = -0.145309i \\
\hat{X}_3 &= \frac{1}{5}(e^{6i\omega_0} + e^{3i\omega_0} - e^{-3i\omega_0} - e^{-6i\omega_0}) = \frac{2i}{5}(\sin(\frac{12\pi}{5}) + \sin(\frac{6\pi}{5})) \\
&= \frac{2i}{5}(-\sin(\frac{8\pi}{5}) - \sin(\frac{4\pi}{5})) = -\hat{X}_2 = 0.145309i \\
\hat{X}_4 &= \frac{1}{5}(e^{8i\omega_0} + e^{4i\omega_0} - e^{-4i\omega_0} - e^{-8i\omega_0}) = \frac{2i}{5}(\sin(\frac{16\pi}{5}) + \sin(\frac{8\pi}{5})) \\
&= \frac{2i}{5}(-\sin(\frac{4\pi}{5}) - \sin(\frac{2\pi}{5})) = -\hat{X}_1 = -0.615537i.
\end{aligned}$$

Therefore, $x(n) = \begin{cases} \hat{X}_1 e^{\frac{2i\pi n}{5}} + \hat{X}_2 e^{\frac{4i\pi n}{5}} + \hat{X}_3 e^{\frac{6i\pi n}{5}} + \hat{X}_4 e^{\frac{8i\pi n}{5}} & -2 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}$ Notice that the DTFS coefficients are odd and purely imaginary. This is also a consequence of conjugate symmetry, since x is real valued and odd, the DTFS coefficients are imaginary valued and odd.

HW8.2 (LTI Processing of Periodic Signals) Consider a periodic, discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the discrete-time Fourier series (DTFS) expansion

$$x[n] = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \,,$$

where ω_0 denotes the fundamental frequency of the signal; if p is the period of x, then $\omega_0 = 2\pi/p$.

Suppose x is the input signal applied to a linear time-invariant (LTI) system characterized by the impulse response $h : \mathbb{Z} \to \mathbb{R}$ and corresponding frequency response H, where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] \, e^{-i\omega n} \,, \quad \forall \omega.$$

Let y be the corresponding output signal.

(a) Let the DTFS expansion of the output signal y be

$$y[n] = \sum_{k=\langle p \rangle} Y_k \, e^{ik\omega_0 n} \,.$$

(i) Express the output-signal DTFS coefficients Y_k in terms of the input signal DTFS X_k and the frequency response H.

Keeping in mind the DTFS expansion of $y[n]$, from above we can read off:

$$Y_k = H(k\omega_0) X_k$$

We now have a relationship between the DTFS coefficients of the input and output from an LTI system H with a given frequency response.

(ii) Suppose the impulse response of the LTI system is given by $h[n] = \delta[n - n_0]$, where $n_0 \in \mathbb{Z}$. Explicitly determine the output-signal DTFS coefficients Y_k in terms of the input-signal DTFS coefficients X_k.

Given $h[n] = \delta[n - n_0]$, we must first determine the corresponding frequency response:

$$H(\omega) = \sum_n h[n] e^{-i\omega n} = \sum_n \delta[n - n_0] e^{-i\omega n} = e^{-i\omega n_0}$$

Finally,

$$Y_k = H(k\omega_0) X_k = e^{-ik\omega_0 n_0} X_k$$

HW8.3 (A DTFS Puzzle) Consider a discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the following the properties:

- $x[n + 4l] = x[n],\ \forall l, n \in \mathbb{Z}$.
- $\displaystyle\sum_{n=-1}^{2} x[n] = 4$.
- $\displaystyle\sum_{n=-1}^{2} (-1)^n x[n] = 8$.
- $\displaystyle\sum_{n=-1}^{2} x[n] \cos\left(\frac{\pi}{2} n\right) = \sum_{n=-1}^{2} x[n] \sin\left(\frac{\pi}{2} n\right) = 0$.

(a) Determine the complex exponential Fourier series coefficients X_{-1}, X_0, X_1, and X_2 for the signal x. From the coefficients, determine and provide a well-labeled plot for the signal x.

More space for part (a):

The DTFS expansion is valid for any period of x, not just its fundamental period. We usually use the fundamental period since it has the least coefficients to compute, but in this case all we know is that x is 4-periodic, so we will compute 4 DTFS coefficients.

Note that many of these hints are projections, or inner products. $\sum_{n=-1}^{2} x[n]$ is just $\langle x, \psi_0 \rangle$ where $\psi_k = e^{i\omega_0 kn}$ (vectorized over the interval of length n). Also remember that the DTFS basis (the roots of unity) are orthogonal; that is, $\langle \psi_a, \psi_b \rangle$ = 0 when $a! = b$, or = p when $a = b$.

Using the synthesis equation, $\langle x, \psi_k \rangle = \langle \sum_j X_j \psi_j, \psi_k \rangle = \langle X_k \psi_k, \psi_k \rangle = X_k \cdot p$.

$$\begin{aligned} x : \mathbb{Z} \to \mathbb{R}, \quad x[n+4l] &= x[n],\ \forall l \in \mathbb{Z} \\ &\Rightarrow\ p = 4/k, \omega_0 = k\pi/2 \text{ for some } k \in \{1, 2, 4\} \end{aligned}$$

We use the DTFS expansion of x assuming $k = 1$, i.e., $p = 4$.

$$\begin{aligned} \sum_{n=-1}^{2} x[n] &= 4 \\ &\Rightarrow \sum_{n=-1}^{2} (X_{-1} e^{-i\frac{\pi}{2}n} + X_0 + X_1 e^{i\frac{\pi}{2}n} + X_2 e^{i\pi n}) = 4 \\ &\Rightarrow \langle x, \psi_0 \rangle = 4 \\ &\Rightarrow 4X_0 = 4 \\ &\Rightarrow X_0 = 1 \end{aligned}$$

We can also take note of the fact that X_0 is always just the average value of the signal over its period, which is $4/4 = 1$.

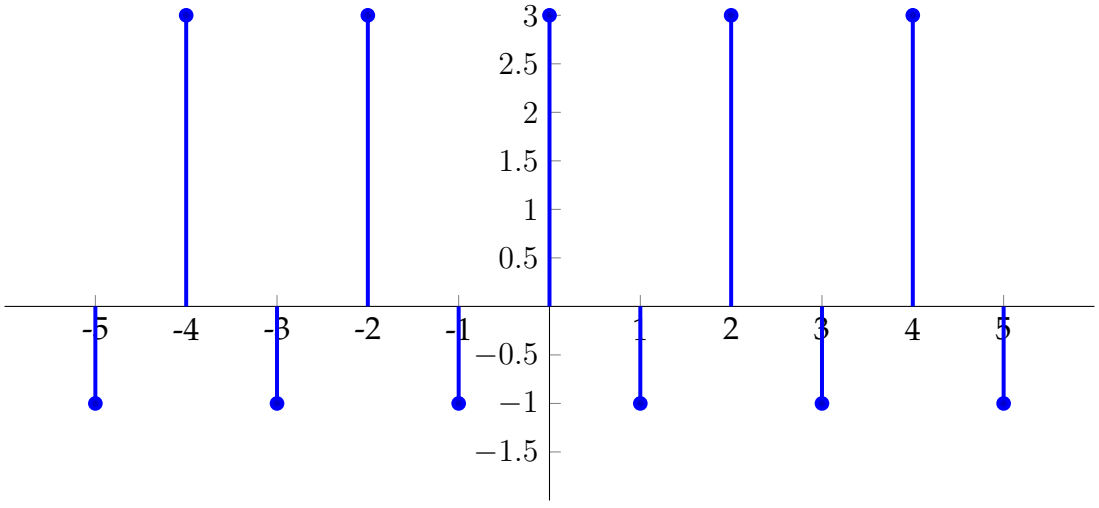
$$\begin{aligned} \sum_{n=-1}^{2} (-1)^n x[n] &= 8 \\ &\Rightarrow \sum_{n=-1}^{2} (-1)^n (X_{-1} e^{-i\frac{\pi}{2}n} + X_0 + X_1 e^{i\frac{\pi}{2}n} + X_2 e^{i\pi n}) = 8 \\ &\Rightarrow \langle x, \psi_2 \rangle = 8 \\ &\Rightarrow 4X_2 = 8 \\ &\Rightarrow X_2 = 2 \end{aligned}$$

$$\begin{aligned}
\sum_{n=-1}^{2} x[n]\cos\left(\frac{\pi}{2}n\right) &= 0 \\
&\Rightarrow \sum_{n=-1}^{2} \frac{1}{2}(e^{i\frac{\pi}{2}n} + e^{-i\frac{\pi}{2}n})(X_{-1}e^{-i\frac{\pi}{2}n} + X_0 + X_1 e^{i\frac{\pi}{2}n} + X_2 e^{i\pi n}) = 0 \\
&\Rightarrow \langle x, 0.5\psi_1 + 0.5\psi_{-1}\rangle = 0 \\
&\Rightarrow 0.5\langle x, \psi_1\rangle + 0.5\langle x, \psi_{-1}\rangle = 0 \\
&\Rightarrow 2X_{-1} + 2X_1 = 0 \\
&\Rightarrow X_{-1} + X_1 = 0 \\
&\Rightarrow X_{-1} = -X_1 \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=-1}^{2} x[n]\sin\left(\frac{\pi}{2}n\right) &= 0 \\
&\Rightarrow \sum_{n=-1}^{2} \frac{1}{2i}(e^{i\frac{\pi}{2}n} - e^{-i\frac{\pi}{2}n})(X_{-1}e^{-i\frac{\pi}{2}n} + X_0 + X_1 e^{i\frac{\pi}{2}n} + X_2 e^{i\pi n}) = 0 \\
&\Rightarrow \frac{1}{2j}(\langle x, \psi_1 - \psi_{-1}\rangle) = 0 \\
&\Rightarrow \frac{4}{2j}(X_{-1} - X_1) = 0 \\
&\Rightarrow X_{-1} - X_1 = 0 \\
&\Rightarrow X_{-1} = X_1 \qquad (2)
\end{aligned}$$

Using Equations 3 and 2, we can deduce that X_{-1} and X_1 must both be 0 (by adding or subtracting the two equations). Hence,

$$x[n] = 1 + 2e^{i\pi n}$$



(b) Based on your results from part (a), determine the fundamental period p and fundamental frequency ω_0 of the signal x. Express x in terms of its DTFS coefficients and complex exponentials of appropriate frequencies, and identify each coefficient and its corresponding harmonic frequency.

From our result in part (a), we can see that $x[n]$ repeats every 2 samples and thus has a fundamental period (p) of 2. It's fundamental frequency is: $\omega_0 = \frac{2\pi}{p} = \pi$.

Hence, from the DTFS synthesis equation,

$$x[n] = \sum_{k=0}^{1} X_k e^{i\frac{2\pi kn}{2}} = X_0 + X_1 e^{i\pi n}$$

We've already seen that

$$x[n] = 1 + 2e^{i\pi n}$$

Hence,

$X_0 = 1$, which corresponds to frequency 0.

$X_1 = 2$, which corresponds to frequency π radians.

Note that in this specific question, x turns out to be 2-periodic, and can be expressed as $x[n] = e^{i0n} + 2e^{i\pi n}$. If we express $x[n]$ as a 2-periodic signal, then its 2 DTFS coefficients would be X_0, X_1 which are the coefficients of the exponentials $e^{i0n}, e^{i\pi n}$ respectively. When we express x as a 4-periodic signal, we find that X_1, X_{-1} which correspond to $e^{i\pi/2n}, e^{i3\pi/2n}$ are 0, which makes sense, since we already determined that x only relies on $e^{i0n}, e^{i\pi n}$. This holds generally as well - if we express a signal that is T-periodic as a (kT)-period signal with kT DTFS coefficients, the only the T coefficients corresponding to the T-periodic signal will be nonzero, and the other $(k-1)T$ coefficients will be 0.

HW8.4 (DTFS as a Change of Basis) In this problem, we will build intuition on Fourier transforms by looking at the DTFS through the lens of change of basis.

We chose the DTFS for this problem because discrete, periodic signals lend themselves well to the vector operations many of you have seen in basic linear algebra.

If we look at the space of discrete-time signals with period p, we can represent each signal as a vector $\in \mathbb{R}^p$ consisting of one period of the signal. To be specific, given a signal $x[n]$ with period p, we can write it as a vector

$$\boldsymbol{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix}$$

(a) To familiarize yourself with this vector notation, find the fundamental period p of $x[n] = \cos(\pi n) + \sin\left(\frac{\pi}{2}n\right)$ and write $x[n]$ as a vector, $\boldsymbol{x}$, in $\mathbb{R}^p$.

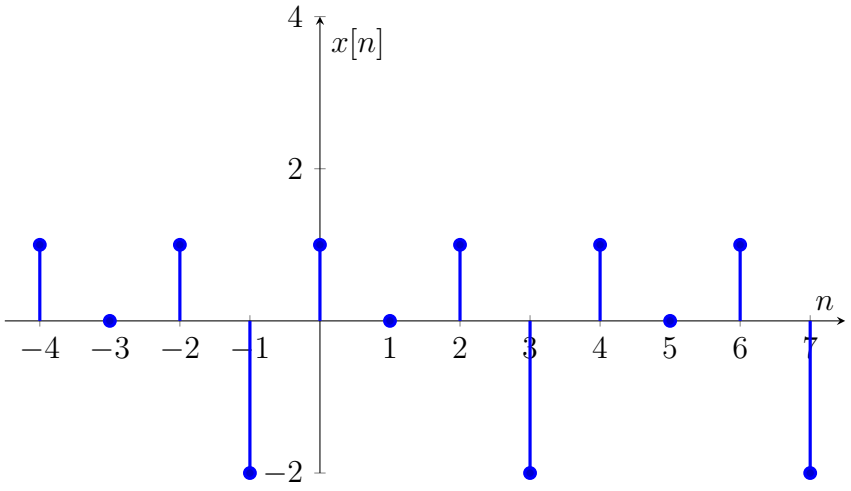
The period of $x[n]$ is the smallest p such that

$$x[n+p] = \cos\left(\pi(n+p)\right) + \sin\left(\frac{\pi}{2}(n+p)\right) = x[n].$$

More specifically, we need to choose the smallest integer p such that πp and $\frac{\pi}{2}p$ are both integer multiples of 2π. So, the fundamental period of $x[n]$ is $p = 4$.

If we evaluate $x[n]$ at every timestep within one period, we get the following vector representation for x:

$$\boldsymbol{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}.$$



(b) We would like to define an orthogonal basis of complex exponentials for $\mathbb{R}^p$ by using the basis signals

$$\psi_k[n] = e^{i\omega_k n}$$

Remember, we are working in the space of p-periodic functions, so $\psi_k[n]$ must correspond to a complex exponential with period p.

Show, for $\omega_k = k\frac{2\pi}{p}$, $\psi_k[n]$ will have a period of p. Explain why only p unique $\boldsymbol{\psi}_k$ exist.

Hint: Think about the roots of unity.

Plugging in ω_k, we get $\psi_k[n] = e^{i\omega_k n} = e^{i\frac{2\pi}{p}n}$, which is p-periodic.

Unlike in continuous time, frequencies in discrete time are 2π-periodic. A result of this is that there are only p unique frequencies of the form $k\frac{2\pi}{p}$. To show why, consider a complex exponential with frequency $\frac{2\pi}{p}$:

$$e^{i(k+p)\frac{2\pi}{p}n} = e^{ik\frac{2\pi}{p}n}e^{2\pi n} = e^{ik\frac{2\pi}{p}n}.$$

This implies that the frequency $(k+p)\frac{2\pi}{p}$ is equivalent to the frequency $k\frac{2\pi}{p}$, so there must only be p unique frequencies of the form $k\frac{2\pi}{p}$.

Intuitively, for example $p = 4$, signals with period of $1, 2, 3, 4$, add up together; the summed signal has period of 4, so for 4-periodic signal space, we need 4 unique basis vectors. $e^{i\omega_k n}$ has a period of p for $\omega_k = k\frac{2\pi}{p}$, $k \in \langle p \rangle$. Note that ω_k corresponds to the frequency of the k^{th} term of the DTFS representation of a p-periodic signal.

(c) Show that the set vectors $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ are orthogonal to each other. A set of vectors $\{\boldsymbol{v}_k\}$ in $\mathbb{R}^p$ is defined as being orthogonal if $\boldsymbol{v}_l^T \boldsymbol{v}_m^* = 0$, $\forall l \neq m$, where the inner product of two complex-valued vectors is defined as follows:

$$\boldsymbol{a}^T \boldsymbol{b}^* = \sum_{n=0}^{p-1} a[n]b^*[n],$$

where $\boldsymbol{b}^*$ is the complex conjugate of $\boldsymbol{b}$ and $b^*[n]$ is the complex conjugate of $b[n]$.

Recall that, in the previous part, we found

$$\omega_k = \frac{2\pi}{p}k, \; k \in \langle p \rangle.$$

Using the definition of orthogonality in the problem statement, let us evaluate the inner product of two arbitrary vectors in the set $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$:

$$\begin{aligned}\boldsymbol{\psi}_l^T \boldsymbol{\psi}_m^* &= \sum_{n=0}^{p-1} e^{i\omega_l n} e^{-i\omega_m n} \\ &= \sum_{n=0}^{p-1} e^{i\frac{2\pi}{p}(l-m)n}.\end{aligned}$$

We can apply the formula for a geometric summation to get

$$\boldsymbol{\psi}_l^T \boldsymbol{\psi}_m^* = \frac{e^{i\frac{2\pi}{p}(l-m)p} - 1}{e^{i\frac{2\pi}{p}(l-m)} - 1}.$$

If $l \neq m$,

$$\boldsymbol{\psi}_l^T \boldsymbol{\psi}_m^* = \frac{e^{i2\pi(l-m)} - 1}{e^{i\frac{2\pi}{p}(l-m)} - 1} = \frac{1-1}{e^{i\frac{2\pi}{p}(l-m)} - 1} = 0.$$

So, the vectors in $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ are orthogonal.

(d) What is the norm of $\boldsymbol{\psi}_k$?

We first find the norm squared of $\boldsymbol{\psi}_k$, and then take the square root to get $||\boldsymbol{\psi}_k||$.

$$||\boldsymbol{\psi}_k||^2 = \boldsymbol{\psi}_k^T \boldsymbol{\psi}_k^* = \sum_{n=0}^{p-1} e^{i\omega_k n} e^{-i\omega_k n} = \sum_{n=0}^{p-1} 1 = p.$$

So, the norm of $\boldsymbol{\psi}_k$ is $\sqrt{p}$.

(e) Because the set of vectors $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ are orthogonal to each other and there are p of them, we can conclude $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ is an orthogonal basis for $\mathbb{R}^p$. Therefore, we can write out any vector $\boldsymbol{x} \in \mathbb{R}^p$ as a linear combination of projections of $\boldsymbol{x}$ onto the vectors in $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$.

Recall

$$\boldsymbol{x} = \sum_k \text{proj}_{\boldsymbol{\psi}_k} \boldsymbol{x} \tag{3}$$

where

$$\text{proj}_{\boldsymbol{\psi}_k} \boldsymbol{x} = \frac{\boldsymbol{x}^T \boldsymbol{\psi}_k^*}{||\boldsymbol{\psi}_k||^2} \boldsymbol{\psi}_k.$$

Show that

$$\text{proj}_{\boldsymbol{\psi}_k} \boldsymbol{x} = X_k \boldsymbol{\psi}_k,$$

where X_k is the k^{th} DTFS coefficient of $x[n]$.

Hint: Use the DTFS analysis equation.

Using the definition of projection provided,

$$\begin{aligned}
\text{proj}_{\boldsymbol{\psi}_k}\boldsymbol{x} &= \frac{\boldsymbol{x}^T\boldsymbol{\psi}_k^*}{||\boldsymbol{\psi}_k||^2}\boldsymbol{\psi}_k \\
&= \left(\frac{1}{||\boldsymbol{\psi}_k||^2}\boldsymbol{x}^T\boldsymbol{\psi}_k^*\right)\boldsymbol{\psi}_k \\
&= \left(\frac{1}{||\boldsymbol{\psi}_k||^2}\sum_{n=0}^{p-1}x[n]\psi_k[n]^*\right)\boldsymbol{\psi}_k \\
&= \left(\frac{1}{p}\sum_{n=0}^{p-1}x[n]e^{-ik\frac{2\pi}{p}n}\right)\boldsymbol{\psi}_k
\end{aligned}$$

Pattern matching with the analysis equation,

$$X_k = \frac{1}{p}\sum_{n=0}^{p-1}x[n]e^{-i\frac{2\pi}{p}kn},$$

we can see that

$$\text{proj}_{\boldsymbol{\psi}_k}\boldsymbol{x} = X_k\boldsymbol{\psi}_k.$$

Recall that $\boldsymbol{\psi}_k = e^{i\omega_0 kn}$ is the k^{th} DTFS exponential.

(f) Show that equation (3) is the DTFS synthesis equation.

Plugging $\text{proj}_{\boldsymbol{\psi}_k}\boldsymbol{x} = X_k\boldsymbol{\psi}_k$ into (3), we have

$$\boldsymbol{x} = \sum_k \text{proj}_{\boldsymbol{\psi}_k}\boldsymbol{x} = \sum_k X_k\boldsymbol{\psi}_k = \sum_k X_k e^{ik\frac{2\pi}{p}n},$$

which matches the DTFS complex exponential representation of $x[n]$.

(g) We now stack X_k into the column vector $\boldsymbol{X} = \begin{bmatrix}X_0 & X_1 & \cdots & X_{p-1}\end{bmatrix}^T$. Using the DTFS representation of $x[n]$, derive the matrix form of the DTFS synthesis equation, also known as the synthesis equation. That is, find $\boldsymbol{\Psi}$, where

$$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{X}$$

You do not have to simplify the complex exponentials in $\boldsymbol{\Psi}$.

Note, from the DTFS synthesis equation taken over the bounds 0 to $p-1$, we

can manipulate the equation to extract $\boldsymbol{X}$.

$$\begin{aligned}
x[n] &= \sum_{k=0}^{p-1} X_k e^{ik\frac{2\pi}{p}n} \\
&= X_0 e^{i0\frac{2\pi}{p}n} + X_1 e^{i1\frac{2\pi}{p}n} + \cdots + X_{p-1} e^{i(p-1)\frac{2\pi}{p}n} \\
&= \begin{bmatrix} e^{i0\frac{2\pi}{p}n} & e^{i1\frac{2\pi}{p}n} & \cdots & e^{i(p-1)\frac{2\pi}{p}n} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{p-1} \end{bmatrix} \\
&= \begin{bmatrix} e^{i0\frac{2\pi}{p}n} & e^{i1\frac{2\pi}{p}n} & \cdots & e^{i(p-1)\frac{2\pi}{p}n} \end{bmatrix} \boldsymbol{X}
\end{aligned}$$

To get $\boldsymbol{x}$, we stack all of the entries like so:

$$\begin{aligned}
\boldsymbol{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} e^{i0\frac{2\pi}{p}0} & e^{i1\frac{2\pi}{p}0} & \cdots & e^{i(p-1)\frac{2\pi}{p}0} \end{bmatrix} \boldsymbol{X} \\ \begin{bmatrix} e^{i0\frac{2\pi}{p}1} & e^{i1\frac{2\pi}{p}1} & \cdots & e^{i(p-1)\frac{2\pi}{p}1} \end{bmatrix} \boldsymbol{X} \\ \vdots \\ \begin{bmatrix} e^{i0\frac{2\pi}{p}(p-1)} & e^{i1\frac{2\pi}{p}(p-1)} & \cdots & e^{i(p-1)\frac{2\pi}{p}(p-1)} \end{bmatrix} \boldsymbol{X} \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} e^{i0\frac{2\pi}{p}0} & e^{i1\frac{2\pi}{p}0} & \cdots & e^{i(p-1)\frac{2\pi}{p}0} \\ e^{i0\frac{2\pi}{p}1} & e^{i1\frac{2\pi}{p}1} & \cdots & e^{i(p-1)\frac{2\pi}{p}1} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i0\frac{2\pi}{p}(p-1)} & e^{i1\frac{2\pi}{p}(p-1)} & \cdots & e^{i(p-1)\frac{2\pi}{p}(p-1)} \end{bmatrix}}_{\boldsymbol{\Psi}} \boldsymbol{X}
\end{aligned}$$

Note we can also express $\boldsymbol{\Psi}$ as

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_0[0] & \psi_1[0] & \cdots & \psi_{p-1}[0] \\ \psi_0[1] & \psi_1[1] & \cdots & \psi_{p-1}[1] \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0[p-1] & \psi_1[p-1] & \cdots & \psi_{p-1}[p-1] \end{bmatrix}$$

(h) Is $\mathbf{\Psi}$ from part (g) an orthogonal matrix? Is it an orthonormal matrix? Why or why not?

Hint: Try to express $\mathbf{\Psi}$ in terms of the vectors $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$, then use parts (c) and (d).

Note that each column corresponds to $\boldsymbol{\psi}_k$.

$$\mathbf{\Psi} = \begin{bmatrix} e^{i0\frac{2\pi}{p}0} & e^{i1\frac{2\pi}{p}0} & \cdots & e^{i(p-1)\frac{2\pi}{p}0} \\ e^{i0\frac{2\pi}{p}1} & e^{i1\frac{2\pi}{p}1} & \cdots & e^{i(p-1)\frac{2\pi}{p}1} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i0\frac{2\pi}{p}(p-1)} & e^{i1\frac{2\pi}{p}(p-1)} & \cdots & e^{i(p-1)\frac{2\pi}{p}(p-1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_0 & \boldsymbol{\psi}_1 & \cdots \boldsymbol{\psi}_{p-1} \end{bmatrix}$$

From parts (c) and (d), we found that $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ are orthogonal and $||\boldsymbol{\psi}_k|| = \sqrt{p}$, so this matrix is orthogonal but not orthonormal.

(i) Using the previous two parts, derive **Parseval's Theorem**:

$$\frac{1}{p}\sum_{n=0}^{p-1}|x[n]|^2 = \sum_{k=0}^{p-1}|X_k|^2 \iff \frac{1}{p}||\boldsymbol{x}||^2 = ||\boldsymbol{X}||^2$$

This relationship directly correlates the "energy" located in the time domain to the energy located in the frequency domain.

Hint: If matrix $\mathbf{A}$ is orthonormal, then $||\mathbf{A}\boldsymbol{x}|| = ||\boldsymbol{x}||$. Is $\mathbf{\Psi}$ orthonormal? If not, can we make it orthonormal?

Method 1: (**Linear Algebra Properties**): We start from $\boldsymbol{x} = \mathbf{\Psi}\boldsymbol{X}$. Since we know $\|\boldsymbol{\psi}_k\| = \sqrt{p}$, we know that $\frac{1}{\sqrt{p}}\mathbf{\Psi}$ would have orthogonal columns with norm 1, so $\frac{1}{\sqrt{p}}\mathbf{\Psi}$ is orthonormal.

Part (g) tells us that $\boldsymbol{x} = \mathbf{\Psi}\boldsymbol{X}$, so

$$\frac{1}{\sqrt{p}}\boldsymbol{x} = \frac{1}{\sqrt{p}}\mathbf{\Psi}\boldsymbol{X}.$$

Taking the norm of both sides gives

$$\left\|\frac{1}{\sqrt{p}}\boldsymbol{x}\right\| = \left\|\frac{1}{\sqrt{p}}\mathbf{\Psi}\boldsymbol{X}\right\|$$
$$\frac{1}{\sqrt{p}}\|\boldsymbol{x}\| = \|\boldsymbol{X}\|$$

Squaring both sides gives $\frac{1}{p}\|\boldsymbol{x}\|^2 = \|\boldsymbol{X}\|^2$.

Method 2: (**Algebra Bash**) We have:

$$\begin{aligned}
\frac{1}{p}\sum_{\langle p\rangle}|x[n]|^2 &= \frac{1}{p}\sum_{n=0}^{p-1}\left|\sum_{k=0}^{p-1}X_k\psi_k[n]\right|^2 \\
&= \frac{1}{p}\sum_{n=0}^{p-1}\left(\sum_{k=0}^{p-1}X_k\psi_k[n]\right)\left(\sum_{k'=0}^{p-1}X_{k'}^*\psi_{k'}[n]^*\right) \\
&= \frac{1}{p}\sum_{k=0}^{p-1}X_kX_k^*\langle\psi_k,\psi_k\rangle \\
&= \frac{1}{p}\sum_{k=0}^{p-1}p|X_k|^2 \\
&= \sum_{k=0}^{p-1}|X_k|^2
\end{aligned}$$

HW8.5 (The DTFS and Its Fraternal Twin, the DFT) To represent finite-length or periodic discrete-time signals, engineers have traditionally used a complex exponential Fourier series expansion that is slightly different from the Discrete-Time Fourier Series (DTFS). The expansion of choice is called the Discrete Fourier Transform (DFT).[1]

In this problem, you will discover the simple relationship between the DTFS and the DFT. Recall that the analysis and synthesis equations of the DTFS expansion of a periodic discrete-time signal $x : \mathbb{Z} \to \mathbb{C}$ are:

$$x[n] \;=\; \sum_{k=\langle p \rangle} X_k \, \psi_k[n] \qquad \overset{\text{DTFS}}{\longleftrightarrow} \qquad X_k \;=\; \frac{\langle x, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle}.$$

Written in the time domain, these equations become:

$$x[n] \;=\; \sum_{k=\langle p \rangle} X_k \, \psi_k[n] \qquad \overset{\text{DTFS}}{\longleftrightarrow} \qquad X_k \;=\; \frac{1}{p} \sum_{n=\langle p \rangle} x[n] \, \psi_k^*[n],$$

where $\psi_k[n] \;=\; e^{\,i\,k\,\omega_0\,n}$, $\langle x, \psi_k \rangle \;=\; \sum_{n=\langle p \rangle} x[n] \; \psi_k^*[n]$, $\langle \psi_k, \psi_k \rangle \;=\; \sum_{n=\langle p \rangle} \psi_k[n] \, \psi_k^*[n]$

(a) Consider the set of basis signals $\Phi = \{\phi_0, \ldots, \phi_{p-1}\}$, where $\phi_k = \frac{1}{p}\,\psi_k$. Determine $\langle \phi_k, \phi_l \rangle$ for the cases $k = l$ and $k \neq l$. Use what you already know about $\langle \psi_k, \psi_l \rangle$ to simplify your work.

$$\begin{aligned} \langle \phi_k, \phi_l \rangle &= \sum_{n=\langle p \rangle} \phi_k[n] \, \phi_l^*[n] \\ &= \frac{1}{p^2} \sum_{n=\langle p \rangle} \psi_k[n] \, \psi_l^*[n] \\ &= \boxed{\frac{1}{p}\,\delta[k-l]}. \end{aligned}$$

[1] The Fast Fourier Transform (FFT) is merely a reference to a family of computationally-efficient algorithms for implementing the DFT. The FFT is not a different transform.

(b) We wish to express a p-periodic signal x in terms of the basis functions in Φ, as follows:

$$x[n] \;=\; \sum_{k=\langle p\rangle} X[k]\;\phi_k[n].$$

Determine an expression for the coefficients $X[k]$ in terms of projections of the signal x onto the basis functions in ϕ. What is the relationship between the DFT coefficients $X[k]$ and the DTFS coefficients X_k? Finally, write down the analysis and synthesis equations for the DFT.

$$\begin{aligned} X[k] &= \frac{\langle x, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle} \\ &= p \sum_{n=\langle p\rangle} x[n]\;\phi_k^*[n] \\ &= \sum_{n=\langle p\rangle} x[n]\;\psi_k^*[n] \\ &= \boxed{p\,X_k}. \end{aligned}$$

Hence,

$$\text{Analysis:}\quad X[k] \;=\; p\,X_k \;=\; \boxed{\sum_{n=\langle p\rangle} x[n]\;e^{-i\,k\,\omega_0\,n}}.$$

$$\text{Synthesis:}\quad x[n] \;=\; \sum_{k=\langle p\rangle} X[k]\;\phi_k[n] \;=\; \boxed{\frac{1}{p}\sum_{k=\langle p\rangle} X[k]\;e^{i\,k\,\omega_0\,n}}.$$

(c) Define $\mathbf{x}$ and $\mathbf{X}_{DFT}$ to be the signal and DFT coefficient vectors, respectively:

$$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix}, \qquad \mathbf{X}_{DFT} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[p-1] \end{bmatrix}.$$

Express the analysis and synthesis equations you found in (b) as a matrix-vector formulation. In other words, find $\mathbf{F}_{DFT}$ and $\mathbf{F}_{iDFT}$ such that

$$\mathbf{X}_{DFT} = \mathbf{F}_{DFT}\,\mathbf{x}, \qquad \mathbf{x} = \mathbf{F}_{iDFT}\,\mathbf{X}_{DFT}.$$

Express $\mathbf{F}_{DFT}$ and $\mathbf{F}_{iDFT}$ in terms of the complex exponential.

For the analysis equation, we assemble $X[k]$ into a vector:

$$\mathbf{X}_{DFT} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[p-1] \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{p-1} x[n]\, e^{-i\,0\,\omega_0\,n} \\ \sum_{n=0}^{p-1} x[n]\, e^{-i\,1\,\omega_0\,n} \\ \vdots \\ \sum_{n=0}^{p-1} x[n]\, e^{-i\,(p-1)\,\omega_0\,n} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{-i\,0\,\omega_0\,0} & e^{-i\,0\,\omega_0\,1} & \dots & e^{-i\,0\,\omega_0\,(p-1)} \\ e^{-i\,1\,\omega_0\,0} & e^{-i\,1\,\omega_0\,1} & \dots & e^{-i\,1\,\omega_0\,(p-1)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-i\,(p-1)\,\omega_0\,0} & e^{-i\,(p-1)\,\omega_0\,1} & \dots & e^{-i\,(p-1)\,\omega_0\,(p-1)} \end{bmatrix}}_{\mathbf{F}_{DFT}} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix}$$

$$= \mathbf{F}_{DFT}\,\mathbf{x}.$$

For the synthesis equation, we assemble $x[n]$ into a vector:

$$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix} = \begin{bmatrix} \frac{1}{p}\sum_{k=0}^{p-1} X[k]\, e^{i\,k\,\omega_0\,0} \\ \frac{1}{p}\sum_{k=0}^{p-1} X[k]\, e^{i\,k\,\omega_0\,1} \\ \vdots \\ \frac{1}{p}\sum_{k=0}^{p-1} X[k]\, e^{i\,k\,\omega_0\,(p-1)} \end{bmatrix}$$

$$= \frac{1}{p}\underbrace{\begin{bmatrix} e^{i\,0\,\omega_0\,0} & e^{i\,0\,\omega_0\,1} & \dots & e^{i\,0\,\omega_0\,(p-1)} \\ e^{i\,1\,\omega_0\,0} & e^{i\,1\,\omega_0\,1} & \dots & e^{i\,1\,\omega_0\,(p-1)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\,(p-1)\,\omega_0\,0} & e^{i\,(p-1)\,\omega_0\,1} & \dots & e^{i\,(p-1)\,\omega_0\,(p-1)} \end{bmatrix}}_{\mathbf{F}_{iDFT}} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[p-1] \end{bmatrix}$$

$$= \mathbf{F}_{iDFT}\,\mathbf{X}_{DFT},$$

where

$$\mathbf{F}_{iDFT} = \frac{1}{p}\begin{bmatrix} e^{i\,0\,\omega_0\,0} & e^{i\,0\,\omega_0\,1} & \cdots & e^{i\,0\,\omega_0\,(p-1)} \\ e^{i\,1\,\omega_0\,0} & e^{i\,1\,\omega_0\,1} & \cdots & e^{i\,1\,\omega_0\,(p-1)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\,(p-1)\,\omega_0\,0} & e^{i\,(p-1)\,\omega_0\,1} & \cdots & e^{i\,(p-1)\,\omega_0\,(p-1)} \end{bmatrix}.$$

Notice that as with the DTFS, we have $\mathbf{F}_{DFT} = p\left(\mathbf{F}_{iDFT}\right)^H$ up to a constant factor (the only difference is multiplying by p instead of $\frac{1}{p}$).