Problem Set 9
Due: Fri, 6 Nov 2026, 23:59
Accepted Thru: Sun, 8 Nov 2026, 23:59

Your LAST Name _____ **Your FIRST Name** _____

Your Collaborators

- Name: _____ SID: _____
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Policy Statement

- We encourage you to collaborate, but only in a group of up to *five* current EE 120 students.
- Be sure to write the names of your collaborators at the appropriate location on the next page. Each collaborator must submit a distinct, self-prepared solution document containing original contributions to the collaborative effort.
- To submit your work, you have several options:
  (a) Print out this homework document and write your solutions on the printout. Then upload a *PDF file* of the scan.
  (b) Write your solutions on this PDF file using a tablet device. Then upload a *PDF file*.
  (c) Use your own sheets of paper, but demarcate clearly the space for each problem, part, and subpart. Then upload a *PDF file*.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.* Remember that, even though the problem sets are self-graded, we will evaluate your work whenever we check your self grades.
- Unless we explicitly state otherwise, you may expect full credit *only if* you explain your work succinctly, but clearly and convincingly.
- If you are asked to provide a "sketch," it refers to a well-labeled *hand-drawn* sketch—not a plot generated by a computing device.
- On occasion, a problem set contains one or more problems designated as "optional." Your solutions to such problems will NOT be evaluated. Nevertheless, you're responsible for learning the subject matter within their scope.
- If you have a confirmed disability that precludes you from complying fully with these instructions or with any other parameter associated with this problem set, please alert us immediately about reasonable accommodations afforded to you by the DSP Office on campus.

Overview

This homework covers:

- CTFS
- DTFT

HW9.1 (A CTFS Puzzle) FourierBot knows the following clues about a periodic, continuous-time signal $x(t)$ and its CTFS coefficients X_k:

(i) $x(t) = x(t+4)$ for all $t \in \mathbb{R}$.

(ii) For $0 \leq k < 4$, there exists $t \in \mathbb{R}$ such that $x(t) \neq x(t+k)$.

(iii) $X_k = X_{-k}^*$ for all $k \in \mathbb{N}$.

(iv) $X_k = 0$ for $k > 2$.

(v) The average value of $x(t)$ over one period is 4.

(vi) $\int_{-3}^{1} |x(t)|^2 dt = 128$.

(vii) $x(t)$ is even.

(viii) $x(0) = 12$.

Determine their signal $x(t)$ **in terms of sines and cosines** (with a potential constant term).

The hints tell us that:

(i) A period of $x(t)$ is 4.

(ii) The fundamental period of $x(t)$ is 4.

(iii) By the conjugate symmetry property, $x(t)$ is real-valued.

(iv) Since $\forall k > 2, X_k = 0$ and we have conjugate symmetry, we also have that $\forall k < -2, X_k = 0$. This means the only nonzero X_k are for $k \in \{-2, -1, 0, 1, 2\}$.

(v) Plugging into the analysis equation using $k = 0$, we have that $X_0 = \frac{1}{p} \int_{\langle p \rangle} x(t) dt$, which equals the average value of $x(t)$ over one period. This means $X_0 = 4$.

(vi) We can apply Parseval's theorem, which tells us that $\frac{1}{p} \int_{\langle p \rangle} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$. This means $\frac{128}{4} = 32 = \sum_{k=-2}^{2} |X_k|^2$.

(vii) Because $x(t)$ is real-valued and even, we know X_k are also real and even. Before we apply this hint further, let's find a general form for $x(t)$. We know $\omega_0 = \frac{2\pi}{p} = \frac{2\pi}{4} = \frac{\pi}{2}$. Since we know the only nonzero X_k are for $k \in \{-2, -1, 0, 1, 2\}$ and that $X_{-1} = X_1$ and $X_{-2} = X_2$, we can write $x(t) = X_2 e^{-i\pi t} + X_1 e^{-i\frac{\pi}{2}t} + X_0 + X_1 e^{i\frac{\pi}{2}t} + X_2 e^{i\pi t}$.

(viii) Plugging in $t = 0$ in the above expression for $x(t)$ gives $12 = 2X_2 + 2X_1 + X_0$. Since we know $X_0 = 4$, we have that $4 = X_2 + X_1$. (eq. 1)

From hints (v) and (vi), we know that $32 = 2|X_2|^2 + 2|X_1|^2 + |X_0|^2$. Subtracting out $X_0^2 = 16$ on both sides gives $16 = 2|X_2|^2 + 2|X_1|^2$, so $8 = |X_2|^2 + |X_1|^2$.

Comparing this result to eq. 1 gives $X_1 = X_2 = 2$. Thus, our final signal is $x(t) = 2e^{-i\pi t} + 2e^{-i\frac{\pi}{2}t} + 4 + 2e^{i\frac{\pi}{2}t} + 2e^{i\pi t}$. Converting this to cosines, we have

$$x(t) = 4 + 4\cos\frac{\pi}{2}t + 4\cos\pi t$$

HW9.2 (Continuous-Time Trigonometric Fourier Series)
A periodic, real-valued continuous-time signal x can be represented by the trigonometric Fourier series expansion

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{+\infty} A_k \cos \frac{2\pi k}{p} t + \sum_{\ell=1}^{+\infty} B_\ell \sin \frac{2\pi \ell}{p} t \\ &= A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{\ell=1}^{+\infty} B_\ell \sin \ell\omega_0 t \,, \end{aligned} \tag{1}$$

where $\omega_0 = \frac{2\pi}{p}$ is the fundamental frequency, and p the fundamental period, of x. Equation 1 is the *synthesis equation* of the trigonometric Fourier series expansion.

More compactly,

$$x = \sum_{k=0}^{+\infty} A_k \chi_k + \sum_{\ell=1}^{+\infty} B_\ell \phi_\ell,$$

which is a linear combination of functions χ_k and ϕ_ℓ, where

$$\begin{aligned} \chi_k(t) &= \cos k\omega_0 t, \quad k \in \mathbb{Z}^+, \text{and} \\ \phi_\ell(t) &= \sin \ell\omega_0 t, \quad \ell \in \mathbb{N}. \end{aligned}$$

Recall that $\mathbb{Z}^+ \triangleq \{0, 1, 2, \ldots\}$ and $\mathbb{N} \triangleq \{1, 2, 3, \ldots\}$.

In this problem, you will show that

$$\{\chi_0, \chi_1, \phi_1, \chi_2, \phi_2, \chi_3, \phi_3, \ldots\}$$

is a set of mutually-orthogonal functions. You will then exploit the mutual orthogonality of these functions to determine the coefficients A_k and B_ℓ.

(a) In what follows you're given a set of inner products, and you're asked to determine a fairly simple expression for each. You may proceed in one of two ways.

In one method, you can express $\chi_k(t)$ and $\phi_\ell(t)$ in terms of the complex exponential functions

$$\psi_r(t) = e^{ir\omega_0 t}, \quad r \in \mathbb{Z},$$

and exploit the orthogonality property of the complex exponentials to arrive at the appropriate expression for each inner product that you're asked to characterize.

Alternatively, you can find each of the inner products in (i)–(iv) below by inserting the appropriate functions in the definition of the inner product. If you choose this method, the following trigonometric identities may prove helpful to you:

$$\begin{aligned} \cos\alpha \cos\beta &= \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta) \\ \cos\alpha \sin\beta &= \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta) \end{aligned}$$

Let $\psi_k(t) = e^{ik\omega_0 t}$, then using the Euler's formula, we can expand $\chi_k(t)$ and $\phi_k(t)$ in terms of $\psi_k(t)$.

$$\chi_k(t) = \frac{1}{2}(e^{ik\omega_0 t} + e^{-ik\omega_0 t}) = \frac{1}{2}(\psi_k(t) + \psi_{-k}(t))$$

$$\phi_k(t) = \frac{1}{2i}(e^{ik\omega_0 t} - e^{-ik\omega_0 t}) = \frac{1}{2i}(\psi_k(t) - \psi_{-k}(t))$$

The orthogonality property of the complex exponential states that $\langle \psi_k, \psi_l \rangle = p\delta(k-l)$ under the inner product defined for continuous time periodic signals. We will be using this property extensively in the following derivations.

(i) $\langle \chi_0, \chi_0 \rangle$.
For the case where k = 0,

$$\chi_0(t) = \frac{1}{2}(\psi_0(t) + \psi_0(t)) = \psi_0(t).$$

Using the orthogonality principle,

$$\langle \chi_0, \chi_0 \rangle = \langle \psi_0, \psi_0 \rangle = p$$

(ii) $\langle \chi_k, \chi_k \rangle$, where $k \neq 0$.
In the case where we take the inner product of χ_k with itself,

$$\begin{aligned}
\langle \chi_k, \chi_k \rangle &= \langle \frac{1}{2}(\psi_k + \psi_{-k}), \frac{1}{2}(\psi_k + \psi_{-k}) \rangle \\
&= \langle \frac{1}{2}\psi_k, \frac{1}{2}\psi_k \rangle + \langle \frac{1}{2}\psi_k, \frac{1}{2}\psi_{-k} \rangle + \langle \frac{1}{2}\psi_{-k}, \frac{1}{2}\psi_k \rangle + \langle \frac{1}{2}\psi_{-k}, \frac{1}{2}\psi_{-k} \rangle \\
&= \frac{1}{4}\langle \psi_k, \psi_k \rangle + \frac{1}{4}\langle \psi_k, \psi_{-k} \rangle + \frac{1}{4}\langle \psi_{-k}, \psi_k \rangle + \frac{1}{4}\langle \psi_{-k}, \psi_{-k} \rangle
\end{aligned}$$

Given that $k \neq 0$, $\langle \psi_k, \psi_{-k} \rangle = \langle \psi_{-k}, \psi_k \rangle = 0$.
This allows us to simplify the expression as follows,

$$\begin{aligned}
\langle \chi_k, \chi_k \rangle &= \frac{1}{4}\langle \psi_k, \psi_k \rangle + \frac{1}{4}\langle \psi_{-k}, \psi_{-k} \rangle \\
&= \frac{p}{4} + \frac{p}{4} = \frac{p}{2}
\end{aligned}$$

(iii) $\langle \chi_k, \chi_m \rangle$, where $k \neq m$.

In the case where we take the inner product of χ_k with χ_m, where $k \neq m \neq 0$, and $k, m \in \mathbb{Z}^+$

$$\begin{aligned}
\langle \chi_k, \chi_m \rangle &= \langle \frac{1}{2}(\psi_k + \psi_{-k}), \frac{1}{2}(\psi_m + \psi_{-m}) \rangle \\
&= \langle \frac{1}{2}\psi_k, \frac{1}{2}\psi_m \rangle + \langle \frac{1}{2}\psi_k, \frac{1}{2}\psi_{-m} \rangle + \langle \frac{1}{2}\psi_{-k}, \frac{1}{2}\psi_m \rangle + \langle \frac{1}{2}\psi_{-k}, \frac{1}{2}\psi_{-m} \rangle \\
&= \frac{1}{4}\langle \psi_k, \psi_m \rangle + \frac{1}{4}\langle \psi_k, \psi_{-m} \rangle + \frac{1}{4}\langle \psi_{-k}, \psi_m \rangle + \frac{1}{4}\langle \psi_{-k}, \psi_{-m} \rangle
\end{aligned}$$

Given that $k \neq m \neq 0$, and $k, m \in \mathbb{Z}^+$
$\langle \psi_k, \psi_{-m} \rangle = \langle \psi_{-k}, \psi_m \rangle = \langle \psi_{-k}, \psi_m \rangle = \langle \psi_{-k}, \psi_{-m} \rangle = 0$.
This allows us to simplify the expression as follows,

$$\langle \chi_k, \chi_m \rangle = 0$$

From this, we can conclude that the basis set $\{\chi_k\}$ also satisfies the orthogonality property.

(iv) $\langle \chi_k, \phi_\ell \rangle$, where $k \in \mathbb{Z}^+$ and $\ell \in \mathbb{N}$.

In the case where we take the inner product of χ_k with ϕ_l, where $k \in \mathbb{Z}^+, l \in \mathbb{N}$,

$$\begin{aligned}
\langle \chi_k, \phi_l \rangle &= \langle \frac{1}{2}(\psi_k + \psi_{-k}), \frac{1}{2i}(\psi_l - \psi_{-l}) \rangle \\
&= \langle \frac{1}{2}\psi_k, \frac{1}{2i}\psi_l \rangle + \langle \frac{1}{2}\psi_k, \frac{-1}{2i}\psi_{-l} \rangle + \langle \frac{1}{2}\psi_{-k}, \frac{1}{2i}\psi_l \rangle + \langle \frac{1}{2}\psi_{-k}, \frac{-1}{2i}\psi_{-l} \rangle \\
&= \frac{-1}{4i}\langle \psi_k, \psi_l \rangle + \frac{1}{4i}\langle \psi_k, \psi_{-l} \rangle + \frac{-1}{4i}\langle \psi_{-k}, \psi_l \rangle + \frac{1}{4i}\langle \psi_{-k}, \psi_{-l} \rangle
\end{aligned}$$

Here we need to look at two different cases:

(1) $k \neq l$ Then following the same reasoning as in (iii),

$$\langle \chi_k, \phi_l \rangle = 0$$

(2) $k = l$

$$\begin{aligned}
\langle \chi_k, \phi_l \rangle &= \frac{-1}{4i}\langle \psi_l, \psi_l \rangle + \frac{1}{4i}\langle \psi_l, \psi_{-l} \rangle + \frac{-1}{4i}\langle \psi_{-l}, \psi_l \rangle + \frac{1}{4i}\langle \psi_{-l}, \psi_{-l} \rangle \\
&= \frac{-1}{4i}\langle \psi_l, \psi_l \rangle + \frac{1}{4i}\langle \psi_{-l}, \psi_{-l} \rangle \\
&= \frac{-p}{4i} + \frac{p}{4i} = 0
\end{aligned}$$

From this, we can conclude that $\langle \chi_k, \phi_l \rangle = 0, \forall k \in \mathbb{Z}^+, \forall l \in \mathbb{N}$.

(b) Take the inner product of each side of Equation 1 with an appropriate function χ_k or ϕ_ℓ, and exploit the mutual orthogonality results you obtained in part (a) to determine the following expressions for the coefficients A_k and B_ℓ, where $k \in \mathbb{Z}^+$ and $\ell \in \mathbb{N}$. These are the *analysis equations* of the trigonometric Fourier series.

$$\begin{aligned} A_0 &= \frac{1}{p}\int_{\langle p\rangle} x(t)dt. \\ A_k &= \frac{2}{p}\int_{\langle p\rangle} x(t)\cos k\omega_0 t\, dt, \quad 1 \le k. \\ B_\ell &= \frac{2}{p}\int_{\langle p\rangle} x(t)\sin \ell\omega_0 t\, dt, \quad 1 \le \ell. \end{aligned}$$

From part a we know that $\langle \phi_l, \chi_0\rangle = \langle \chi_m, \chi_0\rangle = 0$ when $m \neq 0$.

$$\begin{aligned} \int_{\langle p\rangle} x(t)\cdot 1\, dt = \langle x(t), \chi_0\rangle &= \langle \sum_{k=0}^{\infty} A_k\chi_k + \sum_{l=0}^{\infty} B_l\phi_l, \chi_0\rangle \\ &= \langle A_0\chi_0, \chi_0\rangle \\ &= A_0 p \end{aligned}$$

Rearranging, we get:

$$A_0 = \frac{1}{p}\int_{\langle p\rangle} x(t)\, dt$$

Simplifying for A_k:

$$\begin{aligned} \int_{\langle p\rangle} x(t)\cos(k\omega_0 t)\, dt = \langle x(t), \chi_k\rangle &= \langle \sum_{k=0}^{\infty} A_k\chi_k + \sum_{l=0}^{\infty} B_l\phi_l, \chi_k\rangle \\ &= \langle A_k\chi_k, \chi_k\rangle \\ &= A_k\frac{p}{2} \end{aligned}$$

Rearranging, we get:

$$A_k = \frac{2}{p}\int_{\langle p\rangle} x(t)\cos(k\omega_0 t)\, dt \qquad \text{for} \quad k \ge 1$$

Simplifying for B_l:

$$\begin{aligned} \int_{\langle p\rangle} x(t)\sin(l\omega_0 t)\, dt = \langle x(t), \phi_l\rangle &= \left\langle \sum_{k=0}^{\infty} A_k\chi_k + \sum_{l=0}^{\infty} B_l\phi_l, \phi_l\right\rangle \\ &= \langle B_l\phi_l, \phi_l\rangle \\ &= B_l\frac{p}{2} \end{aligned}$$

Rearranging, we get:

$$B_l = \frac{2}{p} \int_{\langle p \rangle} x(t) \sin(l\omega_0 t)\, dt \qquad \text{for} \quad l \geq 1$$

(c) Recall that the complex exponential Fourier series expansion of x is

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k \, e^{ik\omega_0 t},$$

and exploit the fact that x is real-valued (which means that the coefficients X_k are conjugate-symmetric, i.e., $X_k^* = X_{-k}$), to establish the following relations between the trigonometric and complex exponential Fourier series coefficients:

$$\begin{aligned} A_0 &= X_0 \\ A_k &= +2\,\mathsf{Re}(X_k), \quad 1 \le k \\ B_\ell &= -2\,\mathsf{Im}(X_\ell), \quad 1 \le \ell. \end{aligned}$$

We know that:

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{\ell=1}^{+\infty} B_\ell \sin \ell\omega_0 t \\ &= A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{k=1}^{+\infty} B_k \sin k\omega_0 t \\ &= A_0 + \sum_{k=1}^{+\infty} \frac{A_k}{2} e^{ik\omega_0 t} + \sum_{k=1}^{+\infty} \frac{A_k}{2} e^{-ik\omega_0 t} - \sum_{k=1}^{+\infty} i\frac{B_k}{2} e^{ik\omega_0 t} + \sum_{k=1}^{+\infty} i\frac{B_k}{2} e^{-ik\omega_0 t} \\ &= A_0 + \sum_{k=1}^{+\infty} \left(\frac{A_k}{2} - i\frac{B_k}{2} \right) e^{ik\omega_0 t} + \sum_{k=-1}^{-\infty} \left(\frac{A_{-k}}{2} + i\frac{B_{-k}}{2} \right) e^{ik\omega_0 t} \end{aligned}$$

Comparing the above expression to:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} X_k e^{ik\omega_0 t} \\ &= X_0 + \sum_{k=1}^{+\infty} X_k e^{ik\omega_0 t} + \sum_{k=-1}^{-\infty} X_k e^{ik\omega_0 t} \end{aligned}$$

and matching the corresponding terms, we see that $X_0 = A_0$, and $X_k = \frac{A_k}{2} - i\frac{B_k}{2}$ for $k \ge 1$.

Since $x(t)$ is a real signal, then A_k and B_k (or B_ℓ) must also be real. X_k is complex, which implies that:

$X_k = \text{Re}(X_k) + i\text{Im}(X_k) = \frac{A_k}{2} - i\frac{B_k}{2}$ for $k \geq 1$.

and we see that:

$A_k = 2\text{Re}(X_k)$ and $B_k = -2\text{Im}(X_k)$.

With a substitution of variables, we have $B_\ell = -2\text{Im}(X_\ell)$

for $k \geq 1$ and $\ell \geq 1$.

HW9.3 DTFT Warmup Determine the DTFT or inverse DTFT for the following subproblems.

(a)

$$x(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{-i\omega n} = \sum_{n=-\infty}^{-1} \left(\frac{e^{i\omega}}{2}\right)^{-n} \\ &= \sum_{k=1}^{\infty} \left(\frac{e^{i\omega}}{2}\right)^{k} = \frac{e^{i\omega}}{2} \frac{1}{1 - e^{i\omega}/2} = \frac{1}{2e^{-i\omega} - 1} \end{aligned}$$

(b)

$$x[n] = \begin{cases} n & \text{if } |n| \leq 3 \\ 0 & \text{if } |n| > 3 \end{cases}$$

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n} \\ &= -3e^{-i(-3)\omega} - 2e^{-i(-2)\omega} - e^{-i(-1)\omega} + e^{-i(1)\omega} + 2e^{-i(2)\omega} + 3e^{-i(3)\omega} \\ &= 3\left(e^{-i3\omega} - e^{i3\omega}\right) + 2\left(e^{-i2\omega} - e^{i2\omega}\right) + \left(e^{-i\omega} - e^{i\omega}\right) \\ &= -2i\left(3\sin(3\omega) + 2\sin(2\omega) + \sin(\omega)\right) \end{aligned}$$

(c)

$$X(\omega) = \sum_{k=-\infty}^{\infty} (-1)^k \delta\left(\omega - \frac{\pi}{2}k\right)$$

$X(\omega)$ has a Dirac delta at every integer multiple of $\pi/2$. If we apply the analysis equation over $[-\pi/4, 7\pi/4]$, four Dirac deltas will be "sifted."

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{-\pi/4}^{7\pi/4} \sum_{k=-\infty}^{\infty} (-1)^k \delta\left(\omega - \frac{\pi}{2}k\right) e^{i\omega n} d\omega \\ &= \frac{1}{2\pi} \left((-1)^0 e^{i(0)\,n} + (-1)^1 e^{i(\pi/2)\,n} + (-1)^2 e^{i(\pi)\,n} + (-1)^3 e^{i(3\pi/2)\,n}\right) \\ &= \frac{1}{2\pi} \left(1 - i^n + (-1)^n - (-i)^n\right) \end{aligned}$$

HW9.4 (DTFT Property Practice) Note that we can break a signal into even and odd signals. For any $x[n]$, we can find its even and odd components with the following equations,

$$x_e[n] = \frac{x[n] + x[-n]}{2} \qquad x_o[n] = \frac{x[n] - x[-n]}{2} \qquad x[n] = x_e[n] + x_o[n]$$

Note that odd signals are zero at timestep zero, so $x_o[0] = 0$ and $x[0] = x_e[0]$.

By linearity, $x_e[n] \overset{\mathcal{F}}{\longleftrightarrow} X_e(\omega) = \frac{X(\omega)+X(-\omega)}{2}$.

It also can be seen that $\text{Re}\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2}$.

For this problem, consider a ***real-valued***, *causal* discrete-time signal x and its Fourier transform X. The real part of the signal's Fourier transform is given by

$$\text{Re}\{X(\omega)\} = 1 + \cos(\omega) - \cos(2\omega).$$

(a) Using DTFT properties, evaluate the integral $\int_{\langle 2\pi \rangle} X(\omega)\, d\omega$.

Hint: How are $X_e(\omega)$ and $\text{Re}\{X(\omega)\}$ related for a real-valued signal?

First, we note that since x is *real-valued*, $\text{Re}\{X(\omega)\}$ corresponds to the Fourier transform of the even component of $x[n]$. That is,

$$x_e[n] = \frac{x[n] + x[-n]}{2} \overset{\mathcal{F}}{\longleftrightarrow} X_e(\omega) = \text{Re}\{X(\omega)\}.$$

We use the complex conjugate property of the DTFT which says that $x[n] \in \mathbb{R}$ implies $X^*(\omega) = X(-\omega)$ to show this.

$$\text{Re}\{X(\omega)\} = \frac{X(\omega) + X^*(\omega)}{2} = \frac{X(\omega) + X(-\omega)}{2} \overset{\mathcal{F}}{\longleftrightarrow} \frac{x[n] + x[-n]}{2}$$

Next, recall the synthesis equation.

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) \mathrm{e}^{i\omega n} d\omega$$

We find expressions for $x[0]$ and $x_e[0]$.

$$x[0] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{i\omega 0} d\omega$$

$$x_e[0] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X_e(\omega) e^{i\omega 0} d\omega = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} \text{Re}\{X(\omega)\} e^{i\omega 0} d\omega$$

Since

$$x_e[0] = \frac{x[0] + x[0]}{2} = x[0],$$

we can equate the integrals to get

$$\begin{aligned}
\int_{\langle 2\pi \rangle} X(\omega)\, d\omega &= \int_{\langle 2\pi \rangle} \text{Re}\{X(\omega)\}\, d\omega \\
&= \int_{\langle 2\pi \rangle} (1 + \cos(\omega) - \cos(2\omega))\ d\omega \qquad \text{(2a)} \\
&= \int_{\langle 2\pi \rangle} d\omega \qquad \text{(2b)} \\
&= 2\pi
\end{aligned}$$

The step from (2a) to (2b) is easy to make because the integral of a sinusoid over an integer multiple of its period is zero. Here, we're integrating $\cos(\omega)$ over one period (of 2π) and $\cos(2\omega)$ over two periods (of π), so both terms vanish.

(b) Determine, and provide a well-labeled plot of, the signal x.

As mentioned previously, since x is real, we know

$$x_e[n] = \frac{x[n] + x[n]}{2} \overset{\mathcal{F}}{\longleftrightarrow} X_e(\omega) = \text{Re}\{X(\omega)\}.$$

However, in addition to being real-valued, we know that x is causal, so $x[n] = 0,\ \forall n < 0$. Thus, we can conclude:

$$\begin{aligned}
x_e[0] &= x[0] \\
x_e[n] &= \frac{x[n] + x[-n]}{2} = \frac{x[n]}{2} \quad \forall n > 0
\end{aligned}$$

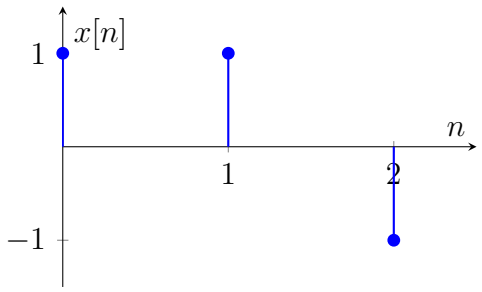
In other words, $x[n] = 2x_e[n],\ \forall n > 0$, and we can find $x[n]$ by finding $x_e[n]$.

$$\begin{aligned}
X_e(\omega) &= \text{Re}\{X(\omega)\} \\
&= 1 + \cos(\omega) - \cos(2\omega) \\
&= 1 + \frac{1}{2}(e^{i\omega} + e^{-i\omega}) - \frac{1}{2}(e^{i2\omega} + e^{-i2\omega})
\end{aligned}$$

By inspection, we see

$$x_e[n] = \delta[n] + \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n+2] - \frac{1}{2}\delta[n-2].$$

From this, we can determine that the non-zero values of $x[n]$ are $x[0] = x_e[0] = 1$, $x[1] = 2x_e[1] = 1$, and $x[2] = 2x_e[2] = -1$. A plot of $x[n]$ is given below.



Alternative Method:

We know that

$$\begin{aligned} X(\omega) &= \sum_{n\in\mathbb{Z}} x[n]e^{-i\omega n} \\ &= \sum_{n\in\mathbb{Z}} x[n]\left[\cos(\omega n) - i\sin(\omega n)\right] \\ &= \underbrace{\left[\sum_{n\in\mathbb{Z}} x[n]\cos(\omega n)\right]}_{\mathrm{Re}\{X(\omega)\}} + i\underbrace{\left[-\sum_{n\in\mathbb{Z}} x[n]\sin(\omega n)\right]}_{\mathrm{Im}\{X(\omega)\}}. \end{aligned}$$

Since $\mathrm{Re}\{X(\omega)\} = 1+\cos(\omega)-\cos(2\omega)$ for all ω, we can simply glean the signal values $x[n]$ from the coefficients of the cosine terms in $\mathrm{Re}\{X(\omega)\}$ (including $\cos(\omega\cdot 0)$). In particular,

$$\mathrm{Re}\{X(\omega)\} = \underbrace{1}_{x[0]}\cdot\cos(\omega\cdot 0) + \underbrace{1}_{x(1)}\cdot\cos(\omega\cdot 1) + \underbrace{(-1)}_{x(2)}\cdot\cos(\omega\cdot 2),$$

and all the other signal values are zero; that is $x[n] = 0$ for $n \neq 0, 1, 2$.

(c) Evaluate the integral $\int_{\langle 2\pi \rangle} X(\omega)\ \cos(\omega)\, d\omega$.

$$\begin{aligned}\int_{\langle 2\pi \rangle} X(\omega)\ \cos(\omega)\, d\omega &= \int_{\langle 2\pi \rangle} X(\omega)\, \frac{1}{2}\left(\mathrm{e}^{i\omega} + \mathrm{e}^{-i\omega}\right)\, d\omega \\ &= 2\pi\, \frac{1}{2}\, \left(x[1] + x[-1]\right) \\ &= \pi\end{aligned}$$

(d) Evaluate the integral $\int_{\langle 2\pi \rangle} |X(\omega)|^2\, d\omega$.

Using Parseval's relation, we have

$$\begin{aligned}\int_{\langle 2\pi \rangle} |X(\omega)|^2\, d\omega &= 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ &= 2\pi \left(|x[0]|^2 + |x[1]|^2 + |x[2]|^2\right) \\ &= 6\pi\end{aligned}$$

(e) Determine a reasonably simple expression for $\mathrm{Im}\{X(\omega)\}$, the imaginary part of the signal's Fourier transform, where, as you know,

$$X(\omega) = \mathrm{Re}\{X(\omega)\} + i\, \mathrm{Im}\{X(\omega)\}.$$

Taking the DTFT of $x[n]$, we have

$$\begin{aligned}X(\omega) &= 1 + e^{-i\omega} - e^{-i2\omega} \\ &= 1 + \cos(\omega) - i\sin(\omega) - \cos(2\omega) + i\sin(2\omega) \\ &= (1 + \cos(\omega) - \cos(2\omega)) + i(-\sin(\omega) + \sin(2\omega))\end{aligned}$$

Thus,

$$\mathrm{Im}\{X(\omega)\} = -\sin(\omega) + \sin(2\omega).$$

Alternatively, we know that for real-valued x, the DTFT of the odd component of x corresponds to $i\mathrm{Im}\{X(\omega)\}$, that is,

$$x_o[n] = \frac{x[n] - x[-n]}{2} \overset{\mathcal{F}}{\longleftrightarrow} X_o(\omega) = i\mathrm{Im}\{X(\omega)\}.$$

We see that $x_o[n] = \frac{1}{2}\delta[n+2] - \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n-2]$, from which we can determine $X_o(\omega)$ and simplify to obtain the same expression for $\mathrm{Im}\{X(\omega)\}$ as given above.

HW9.5 (Reversing the DTFT) Consider a ***real-valued***, *causal*[1] discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having sample values $x[n]$. The discrete-time Fourier transform (DTFT) of x is given by the function $X : \mathbb{R} \to \mathbb{C}$ whose values are denoted by $X(\omega)$.

The following parts are independent. Information given in one part should not be assumed to hold for the others.

(a) The magnitude-squared of X, denoted by $|X(\omega)|^2$, is given by

$$|X(w)|^2 = \frac{5}{4} - \cos\omega \ .$$

Determine a choice for the signal x. Is your answer unique? If so, explain why. If not, find multiple signal that satisfies the property given above.

Hint: Given a complex-valued function $f(x)$, we know that $|f(x)|^2 = f(x)f^*(x)$.
Hint: Think about how to factor the above expression. Start this problem by converting $\cos\omega$ into complex exponentials and distributing each exponential to each different term.

Since $|X(\omega)|^2 = X(\omega)X^*(\omega)$, we attempt to factor the expression into a complex number and its conjugate.

$$\begin{aligned}\frac{5}{4} - \cos\omega &= \frac{5}{4} - \frac{1}{2}\left(e^{i\omega} + e^{-i\omega}\right) \\ &= \frac{1}{4} - \frac{1}{2}e^{i\omega} - \frac{1}{2}e^{-i\omega} + 1 \\ &= \left(\frac{1}{2} - e^{i\omega}\right)\left(\frac{1}{2} - e^{-i\omega}\right)\end{aligned}$$

By negating both factors, we can also factor the expression into

$$\frac{5}{4} - \cos\omega = \left(e^{i\omega} - \frac{1}{2}\right)\left(e^{-i\omega} - \frac{1}{2}\right)$$

Note for both of these factorizations, each term is the conjugate of the other. Thus, $X(\omega)$ could be $\frac{1}{2} - e^{i\omega}$, $\frac{1}{2} - e^{-i\omega}$, $e^{i\omega} - \frac{1}{2}$, or $e^{-i\omega} - \frac{1}{2}$. Using the DTFT synthesis equation, this corresponds to the signals

- $\frac{1}{2}\delta[n] - \delta[n+1]$
- $\frac{1}{2}\delta[n] - \delta[n-1]$

[1]Technically, it is an abuse of terminology to call a signal "causal"; causality is defined for a *system*, not a *signal*. Recall that the impulse response h of a causal LTI system satisfies the constraint, $h : \mathbb{Z} \to \mathbb{C}, h[n] = 0, \forall n < 0$ (if h corresponds to a discrete-time system), or $h : \mathbb{R} \to \mathbb{C}, h(t) = 0, \forall t < 0$ (if h corresponds to a continuous-time system). Borrowing from this LTI vernacular, we say that a discrete-time signal $x : \mathbb{Z} \to \mathbb{C}$ is causal if $x[n] = 0, \forall n < 0$ and a continuous-time signal $x : \mathbb{R} \to \mathbb{C}$ is causal if $x(t) = 0, \forall t < 0$.

- $-\frac{1}{2}\delta[n] + \delta[n+1]$
- $-\frac{1}{2}\delta[n] + \delta[n-1]$

So the solution is not unique.

In this problem, only the magnitude of the frequency response was constrained leaving the phase ambiguous. So shifting the signal by any number of samples in which the phase changes but the magnitude remains the same would also produce a valid answer making the solution not unique.

(b) The imaginary part of $X(\omega)$, denoted by $\text{Im}\{X(\omega)\}$, is characterized by

$$\text{Im}\{X(w)\} = -\sin(2\omega)\,,$$

and it is further known that

$$X(\omega)|_{\omega=0} = 1\,.$$

Determine the signal x. Is your answer unique? If so, explain why. If not, find another signal that satisfies the properties given above.

Hint: It can be seen that $\text{Im}\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2i}$.

Using $\text{Im}\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2i}$, and that
$X^*(-\omega) = X(\omega)$ for real-valued signals, we get

$$\frac{X(\omega) - X(-\omega)}{2i} = -\sin 2\omega = -\frac{e^{i2\omega} - e^{-i2\omega}}{2i}$$

$$\Rightarrow X(\omega) - X(-\omega) = -e^{2i\omega} + e^{-2i\omega}$$

Taking inverse DTFT, we get

$$x[n] - x[-n] = -\delta[n+2] + \delta[n-2]$$

For $n > 0, \Rightarrow x[n] = -\delta[n+2]+\delta[n-2] = \delta[n-2]$ $[x[-n] = 0$ since x is causal$]$

Using definition of DTFT,

$$\Rightarrow X(\omega) = x[0]e^{i0\omega} + e^{-2i\omega} = x[0] + e^{-2i\omega}$$

$$\therefore X(\omega)|_{\omega=0} = 1 \Rightarrow x[0] = 0$$

$\therefore x[n] = \delta[n-2]$ which is unique (no extra assumptions made)